HELLY'S THEOREM AND ITS RELATIVES[1]

BY

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Prologue: Eduard Helly. Eduard Helly was born in Vienna on June 1, 1884. He studied at the University of Vienna under W. Wirtinger and was awarded the Ph. D. degree in 1907. His next few years included further research and study in Göttingen, teaching in a Gymnasium, and publication of four volumes of solutions to problems in textbooks on geometry and arithmetic. His research papers were few in number but contained several important results. The first paper [1] treated some basic topics in functional analysis. Its "selection principle" has found many applications and is often referred to simply as "Helly's theorem" (see Widder's book on the Laplace transform). The same paper contains also the Helly-Bray theorem on sequences of functions (see Widder's book) and a result on extension of linear functionals which is mentioned in the books of Banach and Riesz-Nagy. His famous theorem on the intersection of convex sets (also commonly called "Helly's theorem") was discovered by him in 1913 and communicated to Radon.

Helly joined the Austrian army in 1914, was wounded by the Russians, and taken as a prisoner to Siberia, where one of his "colleagues" was T. Rado. He returned to Vienna in 1920, and in 1921 was married and appointed Privatdozent at the University of Vienna. Along with his mathematical research and work at the University, he held important positions in the actuarial field and as consultant to various financial institutions. His paper [3] on systems of linear equations in infinitely many variables was a much-quoted study of the subject. The "Helly's theorem" which is of special interest to us here was first published by him in 1923 [4] (after earlier publication by Radon and König), and the extension to more general sets in 1930 [5].

In 1938 the Hellys emigrated with their seven-year-old son to America, where Professor Helly was on the staff of Paterson Junior College, Monmouth Junior College (both in New Jersey), and the Illinois Institute of Technology. He died in Chicago in 1943. For much of the above information we are indebted to his wife Elizabeth (also a mathematician, now Mrs. B. M. Weiss), who resides at present in New York City. Their son Walter (Ph. D. from M. I. T.) is a physicist with the Bell Telephone Laboratories in New York.

1. *Über lineare Funktionaloperationen*, S.-B. Akad. Wiss. Wien **121** (1912), 265-297.

[1] This work was supported in part by a grant from the National Science Foundation, U. S. A. (NSF-G18975). The authors wish to express their appreciation to the National Science Foundation, and also to Lynn Chambard for her expert typing.

2. *Über Reihenentwicklungen nach Funktionen eines Orthogonalsystems*, S.-B. Akad. Wiss. Wien **121** (1912), 1539-1549.

3. *Über Systeme linearer Gleichungen mit unendlich vielen Unbekannten*, Monatsh. Math., **31** (1921), 60-97.

4. *Über Mengen konvexer Körper mit gemeinschaftlichen Punkten*, Jber. Deutsch. Math. Verein. **32** (1923), 175-176.

5. *Über Systeme von abgeschlossenen Mengen mit gemeinschaftlichen Punkten*, Monatsh. Math. **37** (1930), 281-302.

6. *Die neue englische Sterblichkeitsmessung an Versicherten*, Assekuranz Jahrbuch, 1934.

Introduction. A subset C of a (real) linear space is called *convex* if and only if it contains, with each pair x and y of its points, the entire line segment $[x, y]$ joining them. The equivalent algebraic condition is that $\alpha x + (1 - \alpha) y \in C$ whenever $x \in C, y \in C$, and $\alpha \in [0, 1]$. At once from the definition comes the most basic and obvious intersection property of convex sets: *the intersection of any family of convex sets is again a convex set*, though of course the intersection may be empty. The present exposition centers around a theorem setting forth conditions under which the intersection of a family of convex sets cannot be empty. This famous theorem of Eduard Helly may be formulated as follows:

HELLY'S THEOREM. *Suppose $\mathscr{K}$ is a family of at least $n + 1$ convex sets in affine n-space R^n, and $\mathscr{K}$ is finite or each member of $\mathscr{K}$ is compact. Then if each $n + 1$ members of $\mathscr{K}$ have a common point, there is a point common to all members of $\mathscr{K}$.*

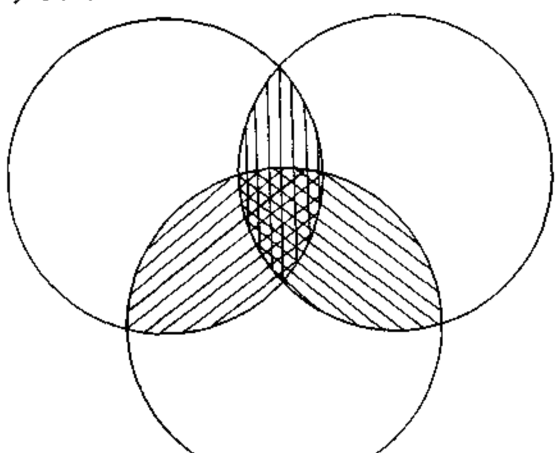


FIGURE 1

Let us inspect two simple examples. Consider first a finite family of compact convex sets in the line R^1, each two of which have a common point. Each set is a bounded closed interval. If $[\alpha_1, \beta_1], \cdots, [\alpha_m, \beta_m]$ are the sets of the

family, it is clear that the point $\min_i \{\beta_i\}$ is common to all of them, as is the point $\max_i \{\alpha_i\}$ and of course all points between these two. Consider next a family of three convex sets in R^2, having a common point (see Figure 1). This gives rise to three shaded areas which are the pairwise intersections of the sets and to a supershaded area which is the intersection of all three. Helly's theorem shows that if a convex set in R^2 intersects each of the three shaded areas, then it must intersect the supershaded area.

The *convex hull* conv X of a set X in a linear space is the intersection of all convex sets containing X. Equivalently, conv X is the set of all *convex combinations* of the points of X. Thus $p \in \text{conv } X$ if and only if there are points $x_1, \cdots, x_m$ of X and positive numbers $\alpha_1, \cdots, \alpha_m$ such that $\sum_1^m \alpha_i = 1$ and $p = \sum_1^m \alpha_i x_i$. Helly's theorem is closely related to the following results on convex hulls:

CARATHÉODORY'S THEOREM. *When $X \subset R^n$, each point of* conv X *is a convex combination of $n + 1$ (or fewer) points of X.*

RADON'S THEOREM. *Each set of $n + 2$ or more points in R^n can be expressed as the union of two disjoint sets whose convex hulls have a common point.*

Consider, for example, a set $X \subset R^2$. By Carathéodory's theorem, each point of conv X must be a point of X, an inner point of a segment joining two points of X, or an interior point of a triangle whose vertices are points of X. If X consists of four points, Radon's theorem implies that one of the points lies in the triangle determined by the other three, or the segment determined by some pair of the points intersects that determined by the remaining pair (see Figure 2).

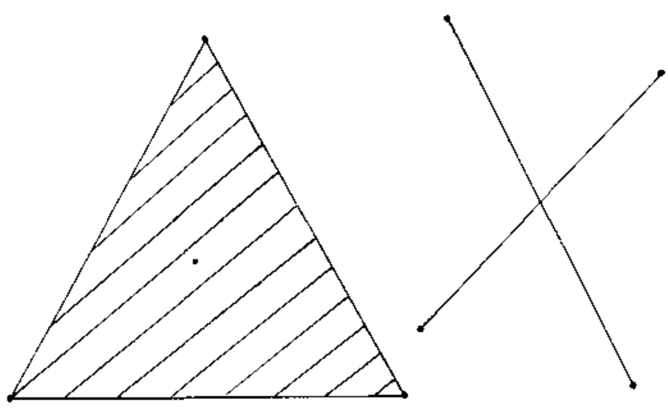


FIGURE 2

The theorem of Carathéodory was published in 1907. Helly's theorem was discovered by him in 1913, but first published by Radon [2] in 1921 (using Radon's Theorem). A second proof was published by König [1] in 1922, and Helly's own proof appeared in 1923 (Helly [1]). Since that time, the three

theorems, and particularly that of Helly, have been studied, applied, and generalized by many authors; especially in the past decade has there been a steady flow of publications concerning Helly's theorem and its relatives. Many of the results in the field (though not always their proofs) would be understandable to Euclid, and most of the proofs are elementary, as is true in most parts of combinatorial analysis which have not been extensively formalized. Some of the results have significant applications in other parts of mathematics, and there are many interesting unsolved problems which appear to be near the surface and perhaps even accessible to the "amateur" mathematician. Thus the study of Helly's theorem and its relatives has several nontechnical aspects in common with elementary number theory, and provides an excellent introduction to the theory of convexity.

The present report is intended to be at once introductory and encyclopedic. Its principal aim is to supply a summary of known results and a guide to the literature. Contents are indicated by section headings, as follows:

Since the report is itself a summary, it seems pointless here to summarize the contents of the individual sections. The emphasis throughout is on combinatorial methods and hence on finite families of subsets of R^n. For intersection properties of infinite families of noncompact convex sets, especially in infinite-dimensional linear spaces, see the report by Klee [**6**].

Many unsolved problems are stated, and after §§1–2 most results are stated without proof. The bibliography of about three hundred items contains all papers known to us which deal with Helly's theorem or its relatives in a finite-dimensional setting. In addition, we list many other papers concerning intersection or covering properties of convex sets, and some general references for the study of convexity and generalized convexities. (We are indebted to E. Spanier and R. Richardson for some relevant references in algebraic topology, and to J. Isbell for information about the paper of Lekkerkerker-Boland [**1**].) Some of the material treated here appears also in books by Bonnesen-Fenchel [**1**], Yaglom-Boltyanskii [**1**], Hadwiger-Debrunner [**3**], Eggleston [**3**], Karlin [**1**], and in the notes of Valentine [**8**]. In general, these will be referred to only for their original contributions. For fuller discussion of some of the unsolved problems mentioned here (and for other elementary problems), see the forthcoming book by Hadwiger-Erdös-Fejes Tóth-Klee [**1**].

Organization of the paper is such that formal designation of various results as lemma, theorem, etc., did not seem appropriate. However, the most important results are numbered, both to indicate their importance and for

purposes of cross-reference. In order to avoid a cumbersome numbering system, we adhere to a convention whereby 4.6 (for example) refers to the numbered result 4.6 itself, 4.6 ff. refers to 4.6 together with immediately subsequent material, 4.6+ refers to material which follows 4.6 but precedes 4.7, and 4.6− refers to material which precedes 4.6 but follows 4.5.

Much of our notation and terminology is commonly used, and should be clear from context. In addition, there is an index to important notions and notations at the end of the paper. Equality by definition is indicated by $:=$ or $=:$. When used in a definition, "provided" means "if and only if"; the latter is also expressed by "iff". The set of all points for which a statement $P(x)$ is true is usually denoted by $\{x: P(x)\}$. However, when $P(x)$ is the conjunction of two statements $P'(x)$ and $P''(x)$, and $P'(x)$ is of especially simple form beginning "$x \cdots$", we sometimes write $\{P'(x): P''(x)\}$ for $\{x: P(x)\}$ in the interest of more natural reading. (For example, $\{x \in E : \|x\| \leqq 1\} := \{x: x \in E$ and $\|x\| \leqq 1\}$.)

Though much of the material is set in an n-dimensional *real linear* space R^n, the full structure of R^n is not always needed. Some of the results are available for finite-dimensional vector spaces over an arbitrary ordered field, while others seem to require that the field be complete or archimedean. We have not pursued this matter. The n-dimensional *Euclidean* space (with its usual metric) is denoted by E^n.

A *convex body* in R^n is a compact convex set with nonempty interior. It is *smooth* provided it admits a unique supporting hyperplane at each boundary point, and *strictly convex* provided its interior contains each open segment $]x, y[$ joining two points of the body. The family of all convex bodies in R^n is denoted by $\mathscr{C}^n$.

A *flat* is a translate of a linear subspace. The group of all translations in R^n is denoted by T^n, or simply by T when there is no danger of confusion. A *positive homothety* is a transformation which, for some fixed $y \in R^n$ and some real $\alpha > 0$, sends $x \in R^n$ into $y + \alpha x$. The group of all positive homotheties in R^n is denoted by H^n (or simply H) and the image of a set X under a positive homothety is called a *homothet* of X.

Set-theoretic intersection, union, and difference are denoted by $\cap$, $\cup$, and $\sim$ respectively, $+$ and $-$ being reserved for vector or numerical sums and differences. The intersection of all sets in a family $\mathscr{F}$ is denoted by $\pi\mathscr{F}$. For a point $x \in R^n$, a real number α, and sets X and $Y \subset R^n$, $\alpha X := \{\alpha x: x \in X\}$, $x + Y := \{x + y : y \in Y\}$, $X + Y := \{x + y : x \in X, y \in Y\}$, TX is the family of all translates of X, and HX is the family of all homothets of X. The interior, closure, convex hull, cardinality, dimension, and diameter of a set X are denoted respectively by $\operatorname{int} X, \operatorname{cl} X, \operatorname{conv} X$, $\operatorname{card} X$, $\dim X$, and $\operatorname{diam} X$. The symbol $\emptyset$ is used for the empty set, 0 for the real number zero as well as for the origin of R^n. The *unit sphere* of a normed linear space is the set $\{x: \|x\| = 1\}$, while its *unit cell* is the set $\{x: \|x\| \leqq 1\}$. The n-dimensional Euclidean unit cell is denoted by B^n, the unit sphere (in E^{n+1}) by S^n.

A *cell* in a metric space (M, ρ) is a set of the form $\{x: \rho(z, x) \leqq \varepsilon\}$ for some $z \in M$ and $\varepsilon > 0$. When we are working with a family HC for a given convex

body C in R^n, these homothets are also called *cells*. When ρ denotes a distance function and X and Y are sets in the corresponding metric space, $\rho(X, Y):= \inf\{\rho(x, y): x \in Y, y \in Y\}$.

In general, points of the space are denoted by small Latin letters, sets by capital Latin letters, families of sets by capital script letters, and properties of families of sets (i.e., families of families of sets) by capital Gothic letters. Small Greek letters are used for real numbers, indices, and cardinalities, and sometimes small Latin letters are also used for these purposes. Variations from this notational scheme are clearly indicated.

1. Proofs of Helly's theorem. As stated in the Introduction, Helly's theorem deals with two types of families: those which are finite and those whose members are all compact. Though the assumptions can be weakened, some care is necessary to exclude such families as the set of all intervals $]0, \beta]$ for $\beta > 0$ or the set of all half-lines $[\alpha, \infty[\subset R^1$. For a family of compact convex sets, the theorem can be reduced at once to the case of finite families, for if the result be known for finite families then in the general case we are faced with a family of compact sets having the *finite intersection property* (that is, each finite subfamily has nonempty intersection), and of course the intersection of such a family is nonempty. This is typical of the manner in which many of the results to be considered here can be reduced to the case of finite families. The essential difficulties are combinatorial in nature rather than topological, and whenever it seems convenient we shall restrict our attention to finite families. The reader himself may wish to formulate and prove the extensions to infinite families. (See also Klee [6] for intersection properties of infinite families of noncompact convex sets.)

For a finite family of convex sets, Helly's theorem may be reduced as follows to the case of a finite family of (compact) convex polyhedra: Suppose $\mathscr{K}$ is a finite family of convex sets (in some linear space), each $n+1$ of which have a common point. Consider all possible ways of choosing $n+1$ members of $\mathscr{K}$, and for each such choice select a single point in the intersection of the $n+1$ sets chosen. Let J be the (finite) set of all points so selected, and for each $K \in \mathscr{K}$ let K' be the convex hull of $K \cap J$. It is evident that each set K' is a convex polyhedron, that each $n+1$ of the sets K' have a common point, and that any point common to all the sets K' must lie in the intersection of the original family $\mathscr{K}$.

We turn now to some of the many proofs of Helly's theorem, with apologies to anyone whose favorite is omitted. (Some other proofs are discussed in §§ 4 and 9.) It seems worthwhile to consider several different approaches to the theorem, for each adds further illumination and in many cases different approaches lead to different generalizations.

Helly's own proof [1] depends on the separation theorem for convex sets and proceeds by induction on the dimension of the space. (Essentially the same proof was given by König [1].) Among the many proofs, this one appeals to us as being most geometric and intuitive. The theorem is obvious for R^0. Suppose it is known for R^{n-1}, and consider in R^n a finite family $\mathscr{K}$ of

at least $n+1$ compact convex sets, each $n+1$ of which have a common point. Suppose the intersection $\pi\mathscr{K}$ is empty. Then there are a subfamily $\mathscr{F}$ of $\mathscr{K}$ and a member A of $\mathscr{F}$ such that $\pi\mathscr{F} = \emptyset$ but $\pi(\mathscr{F} \sim \{A\}) =: M \neq \emptyset$. Since A and M are disjoint nonempty compact convex subsets of R^n, the separation theorem guarantees the existence of a hyperplane H in R^n such that A lies in one of the open halfspaces determined by H and M lies in the other. (To produce H, let $\|\ \ \|$ be a Euclidean norm for R^n, let x and y be points of A and M respectively such that $\|x-y\| = \rho(A, M) := \inf\{\|a-m\|: a \in A, m \in M\}$, and then let H be the hyperplane through the midpoint $(x+y)/2$ which is orthogonal to the segment $[x, y]$. (See Figure 3.) Now let J denote the intersection of some n members of $\mathscr{F} \sim \{A\}$. Obviously $J \supset M$, and since each $n+1$ members of $\mathscr{K}$ have a common point, J must intersect A.

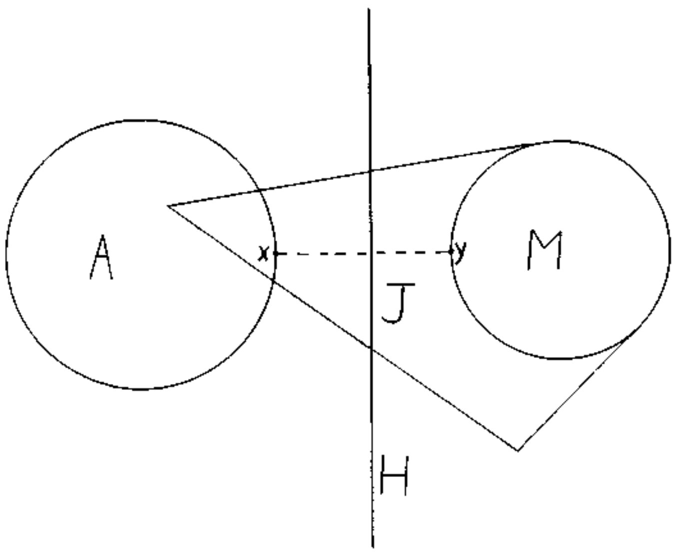


FIGURE 3

Since J is convex, in extending across H from M to A it must intersect H, and thus there is a common point for each n sets of the form $G \cap H$ with $G \in \mathscr{F} \sim \{A\}$. From the inductive hypothesis as applied to the $(n-1)$-dimensional space H it follows that $M \cap H$ is nonempty, a contradiction completing the proof.

Radon's proof [2] is based on the result stated above as Radon's theorem. To prove this result, suppose $p_1, \cdots, p_m$ are points of R^n with $m \geqq n+2$. Consider the system of $n+1$ homogeneous linear equations,

$$\sum_{i=1}^{m} \tau_i = 0 = \sum_{i=1}^{m} \tau_i p_i^j \qquad (1 \leqq j \leqq n),$$

where $p_i = (p_i^1, \cdots, p_i^n)$ in the usual coördinatization of R^n. Since $m > n+1$,

the system has a nontrivial solution $(\tau_1, \cdots, \tau_m)$. Let U be the set of all i for which $\tau_i \geqq 0$, V the set of all i for which $\tau_i < 0$, and $c := \sum_{i \in U} \tau_i > 0$. Then $\sum_{i \in V} \tau_i = -c$ and $\sum_{i \in U} (\tau_i/c) p_i = \sum_{i \in V} (-\tau_i/c) p_i$, completing the proof of Radon's theorem.

Now to prove Helly's theorem for a finite family of convex sets in R^n, we observe first that the theorem is obvious for a family of $n+1$ sets. Suppose the theorem is known for all families of $j-1$ convex sets in R^n, with $j \geqq n+2$, and consider in R^n a family $\mathscr{K}$ of j convex sets, each $n+1$ of which have a common point. By the inductive hypothesis, for each $A \in \mathscr{K}$ there is a point p_A common to all members of $\mathscr{K} \sim \{A\}$, and by Radon's theorem there is a partition of $\mathscr{K}$ into subfamilies $\mathscr{F}$ and $\mathscr{G}$ such that some point z is common to the convex hulls of $\{p_F : F \in \mathscr{F}\}$ and $\{p_G : G \in \mathscr{G}\}$ (see Figure 4, where $\mathscr{F} = \{A, C\}$ and $\mathscr{G} = \{B, D\}$). But then $z \in \pi\mathscr{K}$ and the proof is complete.

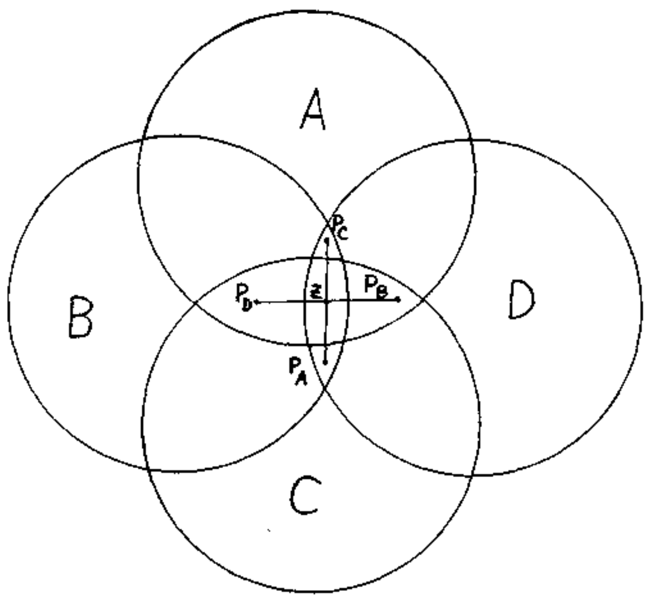


FIGURE 4

The literature contains many other approaches to Helly's theorem. Some of the more interesting may be described briefly as follows:

Rademacher-Schoenberg [1] and Eggleston [3] employ Carathéodory's theorem and a notion of metric approximation.

Sandgren [1] and Valentine [9] employ Carathéodory's theorem and the duality theory of convex cones; Lannér's use of support functionals [1] leads also to application of the duality theory.

Bohnenblust-Karlin-Shapley [1] employ Carathéodory's theorem to prove a result on convex functions from which Helly's theorem follows (see 4.5).

Hadwiger [2; 5] obtains Helly's theorem and other results by an application of the Euler-Poincaré characteristic, which he develops in an elementary setting.

Helly [2] proves a topological generalization by means of combinatorial topology (see 4.11).

R. Rado [2] proves an intersection theorem in a general algebraic setting and deduces Helly's theorem as a corollary (see 9.4).

Levi's axiomatic approach [1] is based on Radon's theorem (see 9.3).

Additional proofs of Helly's theorem are by Dukor [1], Krasnosselsky [2], Proskuryakov [1], and Rabin [1]. For our taste, the most direct and simple approaches to Helly's theorem are those of Helly and Radon described earlier. However, each of the many proofs throws some light on the theorem and related matters which is not shed by others. As is shown by Sandgren [1] and Valentine [9], the duality theory provides efficient machinery for study of Helly's theorem and its relatives. The approach by means of combinatorial topology leads to many interesting problems but remains to be fully exploited.

We have indicated the close relationship of Helly's theorem to the theorems of Carathéodory and Radon. In fact, each of these three theorems can be derived from each of the others, sometimes with the aid of supporting hyperplanes and sometimes without this aid, and each can be proved "directly" by means of induction on the dimension of the space (with the aid of the support or separation theorem). Perhaps the inter-relationships could best be understood by formulating various axiomatic settings for the theory of convexity, and then studying in each the interdependence of these five fundamental results: Helly's theorem, Carathéodory's theorem, Radon's theorem, existence of supporting hyperplanes at certain points of convex sets, existence of separating hyperplanes for certain pairs of convex sets. Levi [1] makes a small step in this direction, and § 9 describes several generalized convexities from which the problem might be approached.

This seems a good place to state a sort of converse of Helly's theorem, due to Dvoretzky [1]. Let us say that a family of sets has the $\mathfrak{H}_n$*-property* provided the intersection of the entire family is nonempty or there are $n + 1$ or fewer sets in the family which have empty intersection. Helly's theorem asserts that each family of compact *convex* sets in R^n has the $\mathfrak{H}_n$-property. Of course the $\mathfrak{H}_n$-property does not characterize convexity, for a family of nonconvex sets in R^n may have the $\mathfrak{H}_n$-property "by accident" (see Figure 5). However, Dvoretzky's theorem may be regarded as saying that if a family of compact sets in R^n has the $\mathfrak{H}_n$-property *by virtue of the linear structure of its members*, then all the members are convex.

DVORETZKY'S THEOREM. *Suppose* $\{K_\iota : \iota \in I\}$ *is a family of compact sets in* R^n *none of which lies in a hyperplane. Then the following two assertions are equivalent:*

all the sets K_ι *are convex;*

if (for each ι*)* J_ι *is affinely equivalent to* K_ι*, then the family* $\{J_\iota : \iota \in I\}$ *has the* $\mathfrak{H}_n$*-property.*

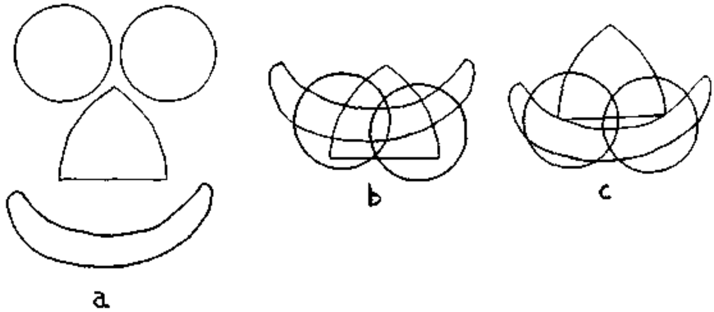


FIGURE 5

An example may be helpful. Each of the families 5a, 5b, and 5c in Figure 5 consists of four sets (all but one convex), obtainable by suitable translations from the members of 5a. The families 5a and 5b have the $\mathfrak{H}_2$-property because of their position rather than by virtue of the linear structure of their members. The family 5c lacks the $\mathfrak{H}_2$-property.

2. Applications of Helly's theorem. We may distinguish two types of applications of Helly's theorem, although the distinction is somewhat artificial. It is used to prove other combinatorial statements of the general form: *If a certain type of collection is such that each of its k-membered subfamilies has a certain property, then the entire collection has the property.* And it is used to prove theorems which are not explicitly combinatorial in statement, but in which a property of a class of sets is established by proving it first for certain especially simple members of the class, and then stepping from this special case to the general result by means of Helly's theorem. The same description applies to applications of Carathéodory's theorem. We shall give several examples of applications of Helly's theorem, with references and comments collected at the end of the section.

2.1. *Suppose $\mathscr{K}$ is a family of at least $n+1$ convex sets in R^n, C is a convex set in R^n, and $\mathscr{K}$ is finite or C and all members of $\mathscr{K}$ are compact. Then the existence of some translate of C which intersects [is contained in; contains] all members of $\mathscr{K}$ is guaranteed by the existence of such a translate for each $n+1$ members of $\mathscr{K}$.*

PROOF. For each $K \in \mathscr{K}$, let $K' := \{x \in R^n : (x + C)rK\}$, where r means "intersects" or "is contained in" or "contains". Then each set K' is convex and the above hypotheses imply that each $n+1$ of the sets K' have a common point. By Helly's theorem, there exists a point $z \in \bigcap_{K \in \mathscr{K}} K'$ and then $(z + C)rK$ for all $K \in \mathscr{K}$.

A *common transversal* for a family of sets is a line which intersects every set in the family.

2.2. *Let $\mathscr{S}$ be a finite family of parallel line segments in R^2, each three of*

which admit a common transversal. Then there is a transversal common to all members of $\mathscr{S}$.

PROOF. We may suppose $\mathscr{S}$ to consist of at least three members and that all segments are parallel to the Y-axis; for each segment $S \in \mathscr{S}$ let C_S denote the set of all points $(\alpha, \beta) \in R^2$ such that S is intersected by the line $y = \alpha x + \beta$. Then each set C_S is convex and each three of such sets have a common point, whence by Helly's theorem there exists a point $(\alpha_0, \beta_0) \in \bigcap_{S \in \mathscr{S}} C_S$. The line $y = \alpha_0 x + \beta_0$ is a transversal common to all members of $\mathscr{S}$.

2.3. *If a convex set in* R^n *is covered by a finite family of open or closed halfspaces, then it is covered by some* $n + 1$ *or fewer of these halfspaces.*

PROOF. Suppose, more generally, that $\mathscr{F}$ is a finite family of sets in R^n covering a convex set C, and that for each $F \in \mathscr{F}$ the set $F' := C \sim F$ is convex. Then $\{F' : F \in \mathscr{F}\}$ is a finite family of convex sets whose intersection is empty, so by Helly's theorem there are $n + 1$ or fewer sets in this family whose intersection is empty. This completes the proof.

2.4. *Two finite subsets* X *and* Y *of* R^n *can be strictly separated (by some hyperplane) if and only if for every set* S *consisting of at most* $n + 2$ *points from* $X \cup Y$, *the sets* $S \cap X$ *and* $S \cap Y$ *can be strictly separated.*

PROOF. We may assume that $X \cup Y$ includes at least $n + 2$ points. For each $x := (x^1, \cdots, x^n) \in X$ and $y := (y^1, \cdots, y^n) \in Y$, let the open halfspaces J_x and Q_y in R^{n+1} be defined as follows:

$$J_x := \left\{\lambda = (\lambda_0, \lambda_1, \cdots, \lambda_n) : \lambda_0 + \sum_1^n \lambda_i x^i > 0\right\},$$

$$Q_y := \left\{\lambda : \lambda_0 + \sum_1^n \lambda_i y^i < 0\right\}.$$

By hypothesis, each $n + 2$ members of the family $\{J_x : x \in X\} \cup \{Q_y : y \in Y\}$ have a common point, and hence by Helly's theorem there exists $\lambda \in (\bigcap_{x \in X} J_x) \cap (\bigcap_{y \in Y} Q_y)$. Then the sets X and Y are strictly separated by the hyperplane $\{z \in R^n : \sum_1^n \lambda_i z^i = -\lambda_0\}$.

When u and v are points of a set $X \subset R^n$, v is said to be *visible* from u (in X) provided $[u, v] \subset X$.

2.5. *Let* X *be an infinite compact subset of* R^n, *and suppose that for each* $n + 1$ *points of* X *there is a point from which all* $n + 1$ *are visible. Then the set* X *is starshaped (that is, there is a point of* X *from which all points of* X *are visible).*

PROOF. For each $x \in X$, let $V_x = \{y : [x, y] \subset X\}$. The hypothesis is that each $n + 1$ of the sets V_x have a common point, and we wish to prove that $\bigcap_{x \in X} V_x \neq \varnothing$. By Helly's theorem, there exists a point $y \in \bigcap_{x \in X} \operatorname{conv} V_x$, and we shall prove that $y \in \bigcap_{x \in X} V_x$. Suppose the contrary, whence there exist $x \in X$ and $u \in [y, x[\sim X$, and there exists $x' \in X \cap [u, x]$ with $[u, x'[\cap X = \varnothing$. Further, there exist $w \in]u, x'[$ such that $\| w - x' \| = (1/2)\rho(\{u\}, X)$, and $v \in [u, w]$

and $x_0 \in X$ such that $\|x_0 - v\| = \rho([u, w], X)$. Since x_0 is a point of X nearest to v, it is evident that V_{x_0} lies in the closed halfspace Q which misses v and is bounded by the hyperplane through x_0 perpendicular to $[v, x_0]$. But then $y \in \operatorname{conv} V_{x_0} \subset Q$ and $\angle yx_0 \geqq \pi/2$, whence $\angle x_0vy < \pi/2$. Since $\rho(\{v\}, X) \leqq \rho(\{w\}, X) < \rho(\{u\}, X)$, it is clear that $v \neq u$ and hence some point of $[u, v[$ is closer to x_0 than v is. This contradicts the choice of v and completes the proof.

The preceding five results have all illustrated the first type of application of Helly's theorem. We now give three applications of the other sort, in which the theorem is not apparently of combinatorial nature, but nevertheless Helly's theorem is very useful.

2.6. *If X is a set in E^n with* $\operatorname{diam} X \leqq 2$, *then X lies in a (Euclidean) cell of radius $[2n/(n+1)]^{1/2}$. If X does not lie in any smaller cell, then* $\operatorname{cl} X$ *contains the vertices of a regular n-simplex of edge-length* 2.

PROOF. We present two proofs of this important result. The first is logically simpler and shows how the theorems of Helly and Carathéodory can sometimes substitute for each other in applications. The second is more understandable from a geometric viewpoint. By combining aspects of these two proofs, one can arrive at a third proof which is easily refined to yield the second equality of 6.8 below.

By Helly's theorem (or 2.1), 2.6 can be reduced to the case of sets of cardinality $\leqq n+1$. For consider $X \subset E^n$ with card $X \geqq n+1$, and for each $x \in X$ the cell $B_x := \{y : \|y - x\| \leqq [2n/(n+1)]^{1/2}\}$. If 2.6 is known for sets of cardinality $\leqq n+1$, then each $n+1$ of the sets B_x have a common point, whence $\bigcap_{x \in X} B_x$ is nonempty by Helly's theorem and the desired conclusion follows.

Now suppose $X \subset E^n$ with card $X \leqq n+1$. Let y denote the center of the smallest Euclidean cell B containing X and let $r(X)$ be its radius. With $r := r(X)$, let

$$\{z_0, \cdots, z_m\} := \{x \in X : \|y - x\| = r\},$$

where $m \leqq n$. It is easily verified that $y \in \operatorname{conv}\{z_0, \cdots, z_m\}$ and we assume without loss of generality that $y = 0$, whence $0 = \sum_0^m \alpha_i z_i$ with $\sum_1^m \alpha_i = 1$ and always $\alpha_i \geqq 0$. For each i and j, let $d_{ij} := \|z_i - z_j\| \leqq 2$, whence $d_{ij}^2 = 2r^2 - 2(z_i, z_j)$. For each j,

$$\begin{aligned} 1 - \alpha_j = \sum_{i \neq j} \alpha_i &\geqq \sum_0^m \alpha_i d_{ij}^2/4 \\ &= r^2/2 - \left(\sum_0^m \alpha_i z_i, z_j\right)\Big/2 = r^2/2, \end{aligned}$$

and summing on j (from 0 to $m \leqq n$) leads to the conclusion that $m \geqq (m+1)r^2/2$, whence

$$r^2 \leqq \frac{2m}{m+1} \leqq \frac{2n}{n+1}.$$

Further, equality here implies that $m = n$ and $d_{ij} = 2$ for all $i \neq j$, so the proof is complete.

In the above paragraph, the assumption card $X \leqq n + 1$ (justified by Helly's theorem) was used only to insure that the point $y \in \text{conv } X$ could be expressed as a convex combination of $n + 1$ or fewer points of X. But this is also insured by Carathéodory's theorem, so the above proof could also be based on the latter.

The second proof of 2.6 is by induction on the dimension. For E^1, the theorem is trivial. Suppose it is known for E^{n-1} and consider a set $X \subset E^n$ with card $X \leqq n + 1$. Let y, B, and $r(X)$ be as above. If $y \in \text{conv } Z$ for some $Z \subsetneqq X$, then the proof is completed by the inductive hypothesis, for B is then the smallest cell containing Z. (It is easily proved that if $y \in \text{conv } Z$ and $p \in E^n \sim \{y\}$, there exists $z \in Z$ with $\|z - y\| < \|z - p\|$. This is equivalent to the lemma mentioned at the end of 9.9 below.)

Hence we may assume that card $X = n + 1$ and $y \in \text{int conv } X$. Now let $\mathscr{Z}$ denote the family of all $(n + 1)$-pointed sets $X \subset E^n$ (with diam $X \leqq 2$) for which $r(X)$ is maximal. By compactness, $\mathscr{Z}$ is nonempty, and clearly $X \in \mathscr{Z}$ implies diam $X = 2$. Consider a set $\{x_0, \cdots, x_n\} \in \mathscr{Z}$ and suppose

$$\min \{\|x_i - x_j\| : 0 \leqq i < j \leqq n\} = \|x_0 - x_1\| < 2 .$$

Then it is not difficult to find another point x_0' near x_0 such that $\|x_0 - x_1\| < \|x_0' - x_1\| < 2$, $\|x_0 - x_i\| = \|x_0' - x_i\|$ for $i > 1$, and $r(\{x_0', \cdots, x_n\}) > r(\{x_0, \cdots, x_n\})$. But of course $\{x_0', \cdots, x_n\} \in \mathscr{Z}$ and the contradiction completes the proof.

2.7. *If C is a convex body in R^n, there exists a point $z \in C$ such that for each chord $[u, v]$ of C which passes through z, $\|z - u\|/\|v - u\| \leqq n/(n + 1)$.*

PROOF. For each point $x \in C$, let $C_x := x + n(n + 1)^{-1}(C - x)$. We claim that $\bigcap_{x \in C} C_x \neq \emptyset$, and to prove this it suffices (in view of Helly's theorem) to show that if $x_0, \cdots, x_n$ are points of C, then $\bigcap_0^n C_{x_i}$ includes the point $y := (n + 1)^{-1} \sum_0^n x_i$. This is evident, since for each j it is true that

$$y = x_j + \frac{n}{n + 1}\left(\frac{1}{n}\sum_{i \neq j} x_i - x_j\right) \in x_j + \frac{n}{n + 1}(C - x_j) .$$

Now consider an arbitrary chord $[u, v]$ passing through the point $z \in \bigcap_{x \in C} C_x$. Then $z \in u + n(n + 1)^{-1}([u, v] - u)$, whence $z = u + n(n + 1)^{-1}\lambda(v - u)$ for some $\lambda \in [0, 1]$ and $\|z - u\|/\|v - u\| = \lambda n(n + 1)^{-1} \leqq n(n + 1)^{-1}$, completing the proof.

2.8. *Let $\mathscr{S}$ be an additive family of sets which includes all open halfspaces in R^n and suppose μ is a function on $\mathscr{S}$ to $[0, \infty[$ which satisfies the following conditions:*

a°. *If $X, Y \in \mathscr{S}$, then $\mu(X \cup Y) \leqq \mu(X) + \mu(Y)$;*

b°. *if $X, Y \in \mathscr{S}$ with $X \subset Y$, then $\mu(X) \leqq \mu(Y)$;*

c°. *there is a bounded set $B \subset R^n$ such that $R^n \sim B \in \mathscr{S}$ and $\mu(R^n \sim B) < (n + 1)^{-1}\mu(R^n)$. Then there exists a point $x^* \in R^n$ such that $\mu(J) \geqq (n + 1)^{-1}\mu(R^n)$ for each open halfspace J containing x^*.*

PROOF. We assume without loss of generality that $\mu(R^n) = 1$. Let $\mathscr{J}$ denote the set of all open halfspaces in R^n and for each $J \in \mathscr{J}$ let J' denote the closed halfspace $R^n \sim J$. Let $\mathscr{G}$ denote the set of all $G \in \mathscr{J}$ such that $\mu(G) < (n+1)^{-1}$. Then for any $n+1$ members $G_0, \cdots, G_n$ of $\mathscr{G}$ we see from a^0 that $\mu(\bigcup_0^n G_i) \leqq \sum_0^n \mu(G_i) < 1$, whence $\bigcup_0^n G_i \neq R^n$ and $\bigcap_0^n G'_i \neq \emptyset$. (Similarly, the theorem is trivial if $\mathscr{G}$ has fewer than $n+1$ members.) It follows from Helly's theorem that every finite subfamily of $\{G' : G \in \mathscr{G}\}$ has nonempty intersection. Now with B as in c^0, it is easy to produce a finite subfamily $\mathscr{F}$ of $\mathscr{J}$ such that the intersection $\bigcap_{F \in \mathscr{F}} F'$ is bounded and contains B. For each $F \in \mathscr{F}$ we have $F \subset R^n \sim \bigcap_{F \in \mathscr{F}} F' \subset R^n \sim B$, whence from b^0 and c^0 it follows that $\mu(F) \leqq \mu(R^n \sim B) < (n+1)^{-1}$ and consequently $F \in \mathscr{G}$. Thus $\{G' : G \in \mathscr{G}\}$ is a family of closed sets with the finite intersection property, and each set in the family intersects the compact set $\bigcap_{F \in \mathscr{F}} F'$. Hence there exists a point $x^* \in \bigcap_{G \in \mathscr{G}} G'$ and this completes the proof of 2.8.

* * * * * * * *

From the eight examples above, it can be seen that Helly's theorem is not only one of the most interesting results but also one of the most important tools in the study of finite-dimensional convexity. For other applications of Helly's theorem, see Yaglom and Boltyanskiĭ [1], Rademacher-Schoenberg [1], Hadwiger-Debrunner [1; 3], and also some of the references below.

Theorem 2.1 is due to Vincensini [2] and Klee [2], a related result to Edelstein [1]. It is a generalization of Helly's theorem, since the latter results when C consists of a single point and r means "intersects" or "is contained in". For various covering problems, 2.1 is useful when the family C consists of one-pointed sets. (See 2.6 and 6.2.)

The result 2.2 is due to Santaló [2] and is discussed also by Rademacher-Schoenberg [1]. In a similar manner, Helly's theorem for R^n yields the following "fitting theorem" of Karlin-Shapley [1]: Suppose $\varphi_1, \cdots, \varphi_n$ are real-valued functions on a linear space E; $x_1, \cdots, x_m$ are points of E; $\alpha_1, \cdots, \alpha_m$ are real numbers; and $\varepsilon_1, \cdots, \varepsilon_m$ are real numbers $\geqq 0$. Then the existence of a linear combination of the φ_i's which fits each point (x_i, α_i) within ε_i (i.e., $|\varphi x_i - \alpha_i| \leqq \varepsilon_i$) is guaranteed by the existence of such a fitting for each $n+1$ points (x_i, α_i). The same paper of Karlin and Shapley contains a result similar to 2.3. (For other approximation theorems obtained with the aid of Helly's theorem, see Šnirel'man [1].)

Theorem 2.4 is due to Kirchberger [1] and the above proof to Rademacher-Schoenberg [1]. (See also Shimrat [1].) The original proof, which did not employ Helly's theorem, is nearly 24 pages long.

Theorem 2.5 is due to Krasnosselsky [1] and related results to Molnár [5] and Valentine [8].

Theorem 2.6 originated with Jung [1], while the first proof above is essentially Gustin's reformulation [2] of Verblunsky's proof [1]. Blumenthal-Wahlin [1] were apparently the first to apply Helly's theorem to this sort of problem, which is studied in more detail in § 6. Carathéodory's theorem was employed by Eggleston [3].

Theorem 2.7 was treated by Minkowski [1] for $n = 2$ and $n = 3$, and in the general case by Radon [1]. However, these authors prove more, namely that the centroid of C has the stated property. The above proof by means of Helly's theorem is apparently due to Yaglom-Boltyanskii [1]. Grünbaum (§ 6 of [16]) treats this and related matters in his discussion of Minkowski's measure of symmetry.

Theorem 2.8 was established by B. H. Neumann [1] for R^2 and under more restrictive hypotheses on μ. The above proof is an improvement of Rado's reasoning in [1]. Grünbaum [13] has given a different proof based on Helly's theorem; however, for validity of his reasoning the mass-distribution should be assumed continuous. See Grünbaum [16] for additional references and related results.

3. **The theorems of Carathéodory and Radon.** We referred earlier to the close relationships among the theorems of Carathéodory, Helly, and Radon. As an additional illustration, we include the standard derivation of Carathéodory's theorem from Radon's. Consider a set $X \subset R^n$ and a point $y \in \operatorname{conv} X$. There are points x_i of X and positive numbers α_i such that $\sum_1^m \alpha_i = 1$, $\sum_1^m \alpha_i x_i = y$, and y cannot be expressed as a convex combination of fewer than m points of X. Suppose $m \geqq n + 2$. Then by Radon's theorem there exist numbers $\beta_1, \cdots, \beta_m$, not all zero but with zero sum, such that $\sum_1^m \beta_i x_i = 0$. Let $V = \{i: \beta_i < 0\}$ and let $j \in V$ be such that $\alpha_j/\beta_j \geqq \alpha_i/\beta_i$ for all $i \in V$. Then we have

$$y = \sum_{i=1}^{m} [\alpha_i + (\alpha_j/\beta_j)\beta_i]x_i ,$$

where the coefficients are all $\geqq 0$, their sum is 1, and the coefficient of x_j is 0. Thus y is a convex combination of $m - 1$ points of X. The contradiction shows that $m \leqq n + 1$ and establishes the theorem of Carathéodory.

For other proofs of Carathéodory's theorem, see Carathéodory [1]. Steinitz [1], Rémès [1], Bohnenblust-Karlin-Shapley [1], Eggleston [3], Pták [2], and others. For some recent refinements and related material, see Motzkin [4; 5], Motzkin-Straus [1], and some of the references below.

Carathéodory's theorem has many interesting applications (e.g., 2.6, 4.5, 6.8). Pták [1] applies Carathéodory's theorem in a beautiful proof of Haar's theorem on the uniqueness of best approximations, and other applications in approximation theory are those of Rémès [1], Rademacher-Schoenberg [1], Mairhuber [1], and Motzkin [5]. (See also Šnirel′man [1].)

An especially simple and useful consequence of Carathéodory's theorem is

3.1. *In R^n, the convex hull of a compact set is compact.*

Proof. Let X be a compact subset of R^n and let A denote the set of all points $\alpha := (\alpha_0, \cdots, \alpha_n) \in R^{n+1}$ such that $\sum_0^n \alpha_i = 1$ and always $\alpha_i \geqq 0$. For each point

$$(\alpha, x) := ((\alpha_0, \cdots, \alpha_n), (x_0, \cdots, x_n)) \in A \times X^{n+1} ,$$

let $f(\alpha, x) := \sum_0^n \alpha_i x_i$. Since f is continuous and $A \times X^{n+1}$ is compact, the set $f(A \times X^{n+1})$ is compact. But by Carathéodory's theorem, $f(A \times X^{n+1}) = \operatorname{conv} X$.

The following result is due essentially to Steinitz [1] and has also been

proved (in various forms) by Dines-McCoy [1], C. V. Robinson [2], Gustin [1], Gale [1], and others. The proof below by means of Carathéodory's theorem is due to Valentine and Grünbaum.

3.2. *If a point y is interior to the convex hull of a set $X \subset R^n$, then y is interior to the convex hull of some set of 2n or fewer points of X.*

Proof. To simplify the notation, we assume without loss of generality that y is the origin 0. With $0 \in \operatorname{int} \operatorname{conv} X$, there is of course a finite subset Y of $\operatorname{conv} X$ such that $0 \in \operatorname{int} \operatorname{conv} Y$ and we conclude the existence of a finite set $V \subset X$ with $0 \in \operatorname{int} \operatorname{conv} V$. Let J denote the set of all linear combinations of $n-1$ (or fewer) points of V. Since clearly $J \neq R^n$, there exists a line L through 0 such that $L \cap J = \{0\}$. Let w_1 and w_2 be the two points of L which are boundary points of $\operatorname{conv} V$ and let H_i denote a hyperplane which supports $\operatorname{conv} V$ at w_i. Clearly $0 \in]w_1, w_2[$ and $w_i \in \operatorname{conv}(V \cap H_i)$. By Carathéodory's theorem and the choice of L, w_i can be expressed as a convex combination of some n points $v_1^i, \cdots, v_n^i$ of $V \cap H_i$ but cannot be expressed as a linear combination of fewer than n points of V. It follows that w_i is interior to the set $\operatorname{conv}\{v_1^i, \cdots, v_n^i\}$ relative to H_i, and since $0 \in]w_1, w_2[$ we see that

$$0 \in \operatorname{int} \operatorname{conv}\{v_1^1, \cdots, v_n^1, v_1^2, \cdots, v_n^2\} .$$

This completes the proof.

Rademacher-Schoenberg [1] deduce from Steinitz's theorem a relative of the theorem 2.4 of Kirchberger [1].

Some recent work of Bonnice-Klee [1] exhibits the theorems of Carathéodory and Steinitz as manifestations of the same underlying result. For a subset Z of R^n and an integer j between 0 and n, define the *j-interior* $\operatorname{int}_j Z$ as the set of all points y such that for some j-dimensional flat $F \subset R^n$, y is interior to $Z \cap F$ relative to F. Then of course $\operatorname{int}_0 Z = Z$ and $\operatorname{int}_n Z = \operatorname{int} Z$. The theorem of Bonnice and Klee is as follows:

3.3. *If $X \subset R^n$ and $y \in \operatorname{int}_j \operatorname{conv} X$, then $y \in \operatorname{int}_j \operatorname{conv} Y$ for some set Y consisting of at most* $\max\{2j, n+1\}$ *points of X.*

Now for a positive integer j and a set X in a linear space, let $H_j(X)$ denote the set of all convex combinations of j or fewer elements of X. Then of course $\operatorname{conv} X = \bigcup_{j=1}^{\infty} H_j(X)$. On the other hand, the convex hull $\operatorname{conv} X$ can also be generated by iteration of the operation H_j for fixed $j > 1$. That is, $\operatorname{conv} X = \bigcup_{i=1}^{\infty} H_j^i(X)$, where $H_j^1(X) := H_j(X)$ and (for $i > 1$) $H_j^i(X) := H_j(H_j^{i-1}(X))$. It is natural to ask how many times the operation H_j must be iterated to produce the convex hull of a set in R^n. The case $j \geqq n+1$ is settled by Carathéodory's theorem. The case $j = 2$ was treated by Brunn [1] and in subsequent years by several other authors (Hjelmslev [1], Straszewicz [1], Bonnesen-Fenchel [1], Abe-Kubota-Yoneguchi [1]). However, the question becomes trivial (*modulo* Carathéodory's theorem) in view of the following almost obvious fact:

3.4. $H_j(H_k(X)) = H_{jk}(X)$.

From this it follows (as noted by Bonnice-Klee [1]) that if $X \subset R^n$ and $j_1 j_2 \cdots j_n \geqq n + 1$, then $H_{j_1}(H_{j_2} \cdots (H_{j_n}(X)) \cdots) = \operatorname{conv} X$; conversely, if X is the set of all vertices of an n-simplex and $j_1 j_2 \cdots j_n \leqq n$, then $H_{j_1}(H_{j_2} \cdots (H_{j_n}(X)) \cdots) \neq \operatorname{conv} X$. (Related problems on the generation of affine hulls are much more complicated. See Klee [8].)

Carathéodory's theorem is the best possible in the sense that if the number $n + 1$ is reduced, the theorem is no longer true for *all* sets $X \subset R^n$. However, the theorem can be sharpened when attention is restricted to special classes of subsets of R^n, such as those which are connected (3.5) or have certain symmetry properties (3.6). A set in R^n is said to be *convexly connected* provided there is no hyperplane H in R^n such that the set misses H but intersects both of the open halfspaces determined by H. Results of Fenchel [1], Stoelinga [1], Bunt [1], and Hanner-Rådström [1] may be stated as follows:

3.5. *Suppose the set $X \subset R^n$ is the union of n connected sets or is compact and the union of n convexly connected sets. Then each point of* conv X *is a convex combination of n or fewer points of X.*

As is noted by Hanner-Rådström [1], the conclusion of 3.5 may fail even in R^2 if the set X is assumed merely to be convexly connected and to be bounded *or* closed but not compact. For $X \subset R^2$, let $X^\Delta := H_2(X) \sim H_1(X)$, the set of all points of conv X which do not lie in X or in any line segment joining two points of X. Then for various bounded convexly connected subsets X of R^2, the set X^Δ may have any finite cardinality and it may be countably or uncountably infinite (the latter proved by P. Erdös, using well-ordering). It is easy to produce closed convexly connected subsets X of R^2 for which X^Δ consists of one or two points. Danzer has given a complicated example in which X^Δ consists of three points. Are there other possibilities?

The following result of Fenchel [4] reduces to Carathéodory's theorem when $m = n$:

3.6. *Suppose G is a group of linear isometries of E^n onto itself, m is the dimension of the set M of all G-invariant points of E^n, and X is a subset of E^n which is mapped into itself by each member of G. Then each point of $M \cap$* conv X *lies in a set* $\operatorname{conv} \bigcup_{g \in G} gY$ *for some set Y consisting of at most $m + 1$ points of X.*

It would be interesting to know just how much 3.3 can be sharpened when the set X is subjected to various restrictions. From 3.4 and 3.5 it is evident that $\operatorname{conv} X = H_j^i(X)$ when X is as in 3.5 and $j^i \geqq n - 1$.

We turn now to some results and problems which were inspired by Radon's theorem. For each pair of natural numbers n and r, let $f(n, r)$ denote the smallest k such that every set of k points in R^n can be divided into r pairwise disjoint sets whose convex hulls have a common point. It follows from Radon's theorem that $f(n, 2) = n + 2$. The function f has been studied by R. Rado [6] and Birch [1], whose results are as follows:

3.7. *For each n and r, $f(n, r) \geqq (n + 1)r - n$, with equality when $n = 1$ and $n = 2$. Always $f(n, r) \leqq 2f(n - 1, r) - n$ and $f(n, r) \leqq rn(n + 1) - n^2 - n + 1$.*

The recursive inequality and determination of $f(1, r)$ are due to Rado and the other results to Birch. Birch's argument employs Carathéodory's theorem, a fixed-point theorem, and the result 2.8 on measures in R^n. Combining the recursive inequality with the fact that $f(2, r) = 3r - 2$, we see that always

$$f(n, r) \leqq 2^{n-2} \cdot 3(r-2) + n + 2 \qquad (\text{for } r \geqq 2),$$

a bound which is sometimes better than $rn(n+1) - n^2 - n + 1$. Other minor improvements are possible, but all known results are far from Birch's conjecture that perhaps always $f(n, r) = (n+1)r - n$. The above results imply that $9 \leqq f(3, 3) \leqq 11$ and $13 \leqq f(3, 4) \leqq 17$, but the exact values of $f(3,3)$ and $f(3, 4)$ are unknown.

Several other problems are implicit in the paper of Rado [6], and a result related to one of these was obtained recently by Birch. For natural numbers n and r with $r < n$, let $b(n, r)$ denote the smallest number k (if one exists) such that for each set of k points in R^n there is an r-dimensional flat which contains $r + 1$ of the points and intersects the convex hull of the remaining points. Birch's result and its proof, not previously published, are as follows:

3.8. *For $n \leqq 2r + 1$, $b(n, r) = n + 2$; for $n \geqq 2r + 2$, $b(n, r)$ does not exist.*

PROOF. Suppose first that $n \leqq 2r + 1$ and $x_1, \cdots, x_{n+2}$ are points of R^n. By Radon's theorem, there are real numbers α_i, not all zero but with zero sum, such that $\sum_1^{n+2} \alpha_i x_i = 0$. At least $n - r + 1$ of the α_i's have the same sign, so we may assume that the numbers $\alpha_1, \cdots, \alpha_{n-r+1}$ are all non-negative and at least one is positive. With $s := \sum_1^{n-r+1} \alpha_i > 0$, we have

$$\sum_{i=1}^{n-r+1} (\alpha_i/s)x_i = \sum_{j=n-r+2}^{n+2} (-\alpha_j/s)x_j ,$$

and this point lies simultaneously in the set conv $\{x_i: 1 \leqq i \leqq n - r + 1\}$ and in the smallest flat containing $\{x_j: n - r + 2 \leqq j \leqq n + 2\}$. It follows that $b(n, r) \leqq n + 2$, and a look at the n-simplex shows that $b(n, r) > n + 1$.

For the case $n \geqq 2r + 2$ it suffices to check that if

$$A := \{\alpha, \alpha^2, \alpha^3, \cdots, \alpha^n) : a > 0\} \subset R^n ,$$

then A has the remarkable property that for any $r + 1$ points $x_0, \cdots, x_r$ of A (with $r \leqq n/2 - 1$), the r-flat through $\{x_0, \cdots, x_r\}$ misses the convex hull of $A \sim \{x_0, \cdots, x_r\}$.

An example similar to A above was first given by Carathéodory [2]. It established the following fact, later rediscovered by Gale [3] and Motzkin [3]:

3.9. *For $2 \leqq m \leqq k$ and $2m \leqq n$, R^n contains a convex polyhedron with k vertices such that each m of these vertices determine a face of the polyhedron.*

For further information on *polyhedral graphs*, see the papers by Gale [5], Grünbaum-Motzkin [2], and others listed by them. For an application of 3.9, see 8.2+.

We should mention also another interesting unsolved problem which involves choosing subsets of a finite set in such a way that certain convex hulls

have *empty* intersection. For natural numbers $2 \leqq n < r$, let $h(n,r)$ denote the smallest number k such that in each set of k points in general position in R^n, there is an r-pointed set X which is *convexly independent* (no point of X lies in the convex hull of the remaining points of X). It is obvious that $h(n, n+1) = n+1$ and easy to verify that $h(n, n+2) = n+3$. Erdös-Szekeres [1] mention a proof by E. Makai and P. Turan that $h(2,5) = 9$, and they establish upper bounds for $h(2,r)$ which are far from their conjecture that $h(2,r) = 2^{r-2}+1$. In a later paper (Erdös-Szekeres [2]) they prove that $h(2,r) \geqq 2^{r-2}$.

De Santis's proof [1] of his generalization of Helly's theorem is based on a generalization of Radon's theorem which applies to sets of flats in a linear space. We shall state a slight extension of the latter result, but this requires further notation. For each finite or infinite sequence r_α of integers with $0 \leqq r_1 \leqq r_2 \leqq \cdots$, let $\xi(r_\alpha)$ denote the smallest integer k for which the following is true: whenever F_α is a sequence of flats in a real linear space, each F_i being of deficiency r_i, then the set of all indices i can be partitioned into complementary sets I and J such that the intersection $(\text{conv} \bigcup_{i \in I} F_i) \cap (\text{conv} \bigcup_{j \in J} F_j)$ contains a flat of deficiency k. When no such k exists, define $\xi(r_\alpha) = \infty$. The following is a slight improvement of De Santis's generalization of Radon's theorem.

3.10. *If m is the smallest integer for which $r_m < m$, then $\xi(r_\alpha) = r_{m+1}$. When this fails to apply, $\xi(r_\alpha) = \infty$.*

It may not be obvious that 3.10 is a generalization of Radon's theorem. To see that it is, consider a sequence $x_1, \cdots, x_k$ of points in R^n. Each point may be regarded as a flat of deficiency n, so the corresponding sequence $r_1, \cdots, r_k$ has always $r_i = n$. When $k \geqq n+1$, then m (in the statement of 3.10) is equal to $n+1$, and thus $\xi(r_\alpha) = r_{n+2}$ provided this is defined; otherwise $\xi(r_\alpha) = \infty$. It follows that when $k \geqq n+2$ the set of indices $\{1, \cdots, k\}$ can be partitioned into complementary sets I and J such that the intersection $\text{conv}\{x_i : i \in I\} \cap \text{conv}\{x_j : j \in J\}$ contains a flat of deficiency n; that is, the intersection is nonempty.

By combining the ideas of 3.10 and 3.8, one may obtain further generalizations of Radon's theorem. However, these appear to be of only marginal interest.

4. **Generalizations of Helly's theorem.** This section contains some generalizations and other relatives of Helly's theorem, all in R^n except for a few supplementary comments. (See § 9 for generalizations in other settings.) At the end of the section we formulate some general problems which may serve as a guide to further research in this area. Division of material between § 3 and § 4 is somewhat artificial in view of the close relationships between the theorem of Helly on the one hand and those of Carathéodory and Radon on the other. Nevertheless, it seems to be justified in that the two different lines of investigation are different in spirit and lead to different generalizations.

Helly's theorem may be regarded as saying something about the "structure" of certain families $\mathscr{K}$ of convex sets in R^n for which $\pi\mathscr{K} = \emptyset$. Specifically, it says that if $\mathscr{K}$ is finite or all members of $\mathscr{K}$ are compact, then there is a subfamily $\mathscr{F}$ consisting of at most $n+1$ members of $\mathscr{K}$ such that $\pi\mathscr{F} = \emptyset$. This suggests the attempt to find theorems which say something about the structure of *every* family $\mathscr{K}$ of convex sets in R^n for which $\pi\mathscr{K} = \emptyset$, and which has Helly's theorem as a consequence. The known partial results in this direction are summarized in a separate report by Klee [6], since they seem more akin to the infinite-dimensional considerations. However, the relevant papers are listed also in our bibiliography (Gale-Klee [1], Karlin-Shapley [1], Klee [2; 4], Rado [5], Sandgren [1]).

As applied to certain families $\mathscr{K}$ of convex sets, Helly's theorem asserts the existence of a point x common to all members of $\mathscr{K}$. The set $\{x\}$ may be regarded, for $j = 0$, in any of the following six ways:

$$\text{as a } \begin{Bmatrix} j\text{-dimensional convex set} \\ j\text{-dimensional flat} \\ (j+1)\text{-pointed set} \end{Bmatrix} \text{ which } \begin{Bmatrix} \text{is contained in} \\ \text{intersects} \end{Bmatrix} \text{ each member of } \mathscr{K}.$$

In each of the six cases, we may ask for conditions on a family $\mathscr{K}$ which assure the existence of such sets corresponding to other values of j. Two of these questions are redundant, for there exists a j-dimensional convex set intersecting all members of $\mathscr{K}$ if and only if there exists a j-dimensional flat intersecting all members of $\mathscr{K}$, and (since the members of $\mathscr{K}$ are convex) for $j \geqq 1$ there exists a $(j+1)$-pointed set which is contained in all members of $\mathscr{K}$ if and only if there exists a 1-dimensional convex set which is contained in all members of $\mathscr{K}$. The remaining four questions have all led to generalizations of Helly's theorem. The most recent of these is the following theorem of Grünbaum [18]:

4.1. *Let* $g(n,0) = n+1, g(n,1) = 2n, g(n,j) = 2n - j$ *for* $1 < j < n$, *and* $g(n,n) = n+1$. *If* $\mathscr{K}$ *is a finite family of at least* $g(n,j)$ *convex sets in* R^n *and each* $g(n,j)$ *members of* $\mathscr{K}$ *have an at-least-j-dimensional intersection, then the intersection* $\pi\mathscr{K}$ *is at least j-dimensional.*

When $j = 0$, this is merely Helly's theorem. Examples show that for each choice of j and n (with $0 \leqq j \leqq n$), $g(n,j)$ is the smallest integer which has the stated property.

The following result is due to De Santis [1]:

4.2. *If* $\mathscr{K}$ *is a finite family of at least* $n+1-j$ *convex subsets of* R^n *and the intersection of each* $n+1-j$ *members of* $\mathscr{K}$ *contains a j-dimensional flat, then* $\pi\mathscr{K}$ *contains a j-dimensional flat.*

This also becomes Helly's theorem when $j = 0$, and the integer $n+1-j$ is the smallest which has the stated property. The proof of De Santis is based on an analogue of Radon's theorem (see 3.10), and Valentine [9] has given a proof by means of the duality theory. As De Santis has noted, his theorem

may be stated in the following form which is independent of the dimension of the space: *Suppose $\mathscr{K}$ is a finite family of convex sets in a real linear space E (which may even be infinite-dimensional) and the intersection of each j members of $\mathscr{K}$ contains a flat of deficiency $< j$ in E. Then $\pi\mathscr{K}$ contains a flat of deficiency $< j$.*

Theorems 4.1 and 4.2 are especially satisfactory generalizations of Helly's theorem, since in each an intersection property of the entire family $\mathscr{K}$ follows from the assumption of that same property for certain subfamilies of $\mathscr{K}$. Now, turning to another of the four questions mentioned above, suppose we wish to produce for some $j \geqq 1$ a j-dimensional flat which intersects all members of $\mathscr{K}$. This is the problem of common transversals discussed in §5. Simple examples show that the existence of a transversal common to all members of $\mathscr{K}$ is not assured merely by assuming this for certain subfamilies of $\mathscr{K}$. Indeed, for each $m \geqq 3$ there are m congruent line segments in the plane such that each $m - 1$ of them admit a common transversal but no line intersects all m of them (Santaló [1]). (For other examples see the references cited in §5.) The only true generalization of Helly's theorem in the direction of common transversals is the following result of Horn [1] and Klee [1]:

4.3. *For integers $1 \leqq j \leqq n + 1$ and a family $\mathscr{K}$ of at least j compact convex sets in R^n, the following three assertions are equivalent:*

a^0. *each j members of $\mathscr{K}$ have a common point;*

b^0. *each flat of deficiency $j - 1$ in R^n admits a translate which intersets all members of $\mathscr{K}$;*

c^0. *each flat of deficiency $j - 2$ in R^n lies in a flat of deficiency $j - 1$ which intersects all members of $\mathscr{K}$.*

The two-dimensional case had been considered earlier by Horn-Valentine [1]. For the general case, Horn [1] (and later Karlin-Shapley [1]) proved that a^0 implies c^0, and the rest is due to Klee [1]. Alternate proofs are given by Hadwiger [4] and Valentine [8]. Klee's proof depends on the following description of the "hole" in R^n which is "surrounded" by $j + 1$ open convex sets $C_0, \cdots, C_j$ such that each j have a common point but no point is common to all $j + 1$: *there is a flat F of deficiency j such that the set $\bigcup_0^j C_i$ misses F but intersects every half-flat bounded by F in a flat of deficiency $j - 1$ containing F.*

For $j = n + 1$ each of the conditions b^0 and c^0 above asserts that $\pi\mathscr{K} \neq \emptyset$, and thus Helly's theorem is obtained. For $j = 2 = n$, the theorem asserts that if there is a common point for each two members of a family $\mathscr{K}$ of compact convex sets in R^2, then each line L in R^2 is parallel to some line L' which intersects all members of $\mathscr{K}$ and each point $x \in R^2$ lies on a line X which intersects all members of $\mathscr{K}$ (see Figure 6). Regarding the lines parallel to a given line as the pencil of lines through a certain point at infinity, we see that the real projective space is a natural setting for 4.3. The theorem may also be approached by means of spherical convexity (Horn [1]). (See 9.1 and 9.2 for a discussion of spherical and projective convexity.)

Now consider the fourth of the questions mentioned above: for $j > 1$, what

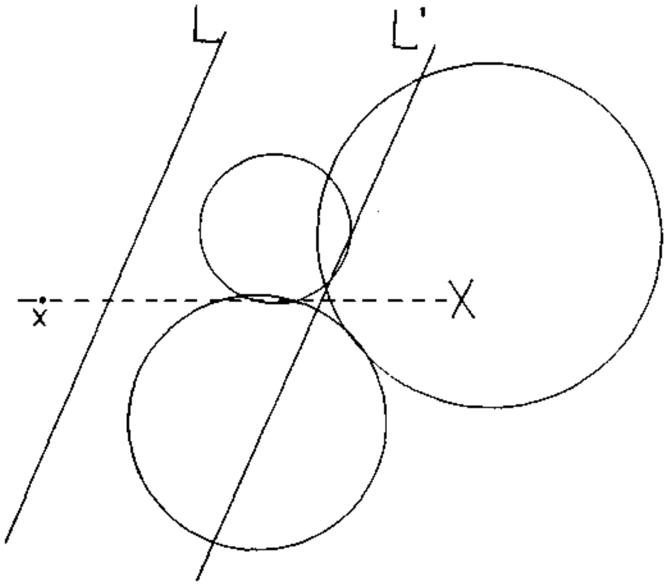


FIGURE 6

conditions on a family $\mathscr{K}$ of convex sets in R^n will assure the existence of a j-pointed set which intersects each member of $\mathscr{K}$? (Cf. the discussion of Gallai-type problems in 7.4+ and near the end of this section.) It is convenient to employ the terminology of Hadwiger-Debrunner [2], in which a *j-partition* of $\mathscr{K}$ is a partition of $\mathscr{K}$ into j subfamilies each of which has nonempty intersection. Under certain stringent geometric restrictions on $\mathscr{K}$, the existence of a j-partition for $\mathscr{K}$ is assured by the existence of partitions for certain subfamilies of $\mathscr{K}$ (see § 7). However, an example of Hadwiger-Debrunner [3] shows that for arbitrarily large integers m there exists a family $\mathscr{K}$ of m plane convex sets such that each proper subfamily of $\mathscr{K}$ admits a 2-partition but $\mathscr{K}$ itself does not. It seems probable that such a family exists for each $m \geqq 3$. Hadwiger-Debrunner [2] approach the problem of j-partitions by means of the following theorem.

4.4. *For $r \geqq j(n-1)+2$, the existence of a j-partition for a finite family $\mathscr{K}$ of convex sets in R^n is implied by the condition that among each $r+j-1$ sets in $\mathscr{K}$, some r have a common point. The same condition does not imply the existence of a $(j-1)$-partition.*

When $j=1$ and $r=n+1$, this reduces to Helly's theorem. The result may be formulated somewhat differently, as follows: for integers r, s, and n, let $J(r, s, n)$ denote the smallest integer j for which a j-partition is admitted by each finite family $\mathscr{K}$ of convex sets in R^n such that among each s members of $\mathscr{K}$, some r have a common point. Then 4.4 asserts that

$J(r, s, n) = s - r + 1$ whenever $r \leqq s$ and $nr \geqq (n-1)s + (n+1)$. It is noted by Hadwiger-Debrunner [2] that $J(r, s, n) = \infty$ when $s \leqq n$ and that always $J(r, s, n) \geqq s - r + 1$. There remains the problem of completely determining the numbers $J(r, s, n)$, which is solved by 4.4 for $n = 1$ but not even for $n = 2$. In fact, the above result yields only $J(4, 3, 2) \geqq 2$, while examples of Danzer (in Hadwiger [12], Hadwiger-Debrunner [3]) and Grünbaum [9] show that $J(4, 3, 2) \geqq 3$. Danzer's example consists of six *congruent* triangles, while Grünbaum's consists of nine *translates* of an arbitrary centrally symmetric strictly convex body in R^2. It is unknown even whether $J(4, 3, 2)$ is finite. (For a generalization of the functions J, see 7.4+.)

Yet another generalization of Helly's theorem arose in connection with the theory of games. It is due to Bohnenblust-Karlin-Shapley [1], and various applications to Karlin-Shapley [1].

4.5. *Suppose C is a compact convex set in R^n and Φ is a finite family of continuous convex functions on C such that for each $x \in C$ there exists $\varphi \in \Phi$ with $\varphi(x) > 0$. Then there are positive numbers $\alpha_0, \cdots, \alpha_j$ with $j \leqq n$ and members $\varphi_0, \cdots, \varphi_j$ of Φ such that $\sum_0^j \alpha_i \varphi_i(x) > 0$ for all $x \in C$.*

To deduce Helly's theorem from 4.5, let $\mathscr{K}$ be a finite family of compact convex sets in R^n, C a compact convex set containing their union, and for each $K \in \mathscr{K}$ and $x \in C$ let $\varphi_K(x) := \rho(\{x\}, K)$. Then if the intersection $\pi\mathscr{K}$ is empty, the set of functions $\Phi := \{\varphi_K : K \in \mathscr{K}\}$ satisfies the hypotheses of 4.5 and hence there exist positive $\alpha_1, \cdots, \alpha_j$ with $j \leqq n$ such that $\sum_0^j \alpha_i \varphi_{K_i}(x) > 0$ for all $x \in C$. This implies that $\bigcap_0^j K_i = \emptyset$ and Helly's theorem follows.

A well-known topological theorem of Knaster-Kuratowski-Mazurkiewicz [1] has the following corollary, of which an elementary proof was given by Klee [1]: *If $j + 1$ closed convex sets in R^n have convex union and each j of them have a common point, then there is a point common to all.* Using this fact as a lemma, Levi [2] proved:

4.6. *Suppose $\mathscr{F}$ is a finite family of at least n closed convex sets in R^n and $\mathscr{F}$ has the following two properties: $\mathfrak{U}_{n+1}$: the union of any $n + 1$ members of $\mathscr{F}$ has connected complement in R^n; $\mathfrak{J}_n$: each n members of $\mathscr{F}$ have a common point.*

Then there is a point common to all members of $\mathscr{F}$.

This may be regarded as a generalization of Helly's theorem, since $\mathfrak{U}_{n+1}$ is obviously implied (for $n > 1$) by the condition that the members of $\mathscr{F}$ are bounded and convex and each $n + 1$ have a common point.

The above lemma was extended in another way by Berge [2] and Ghouila-Houri [1], though their results do not formally imply Helly's theorem. Rather than stating in full the result of Ghouila-Houri [1], we state its two most interesting corollaries as

4.7. *Suppose $C_1, \cdots, C_m$ are closed convex sets in a topological linear space and each k of the sets have a common point, where $1 \leqq k < m$. If $\bigcup_1^m C_i$ is convex, then some $k + 1$ of the sets have a common point. If $k = m - 1$ and*

$\bigcap_1^m C_i = \emptyset$, *then each* $m-1$ *of the sets have a common point in every closed set* X *such that* $X \cup \bigcup_1^m C_i$ *is convex.*

The second part of 4.8, describing the "hole" surrounded by the sets in question, may be compared with the description in 4.3+. The result 4.7 can also be approached by means of the Euler characteristic (Hadwiger [5], Klee [7]).

For a family $\mathscr{F}$ of sets, let us define the *Helly-number* $\alpha(\mathscr{F})$ to be the smallest cardinal k such that whenever $\mathscr{G}$ is a finite subfamily of $\mathscr{F}$ and $\pi\mathscr{S} \neq \emptyset$ for all $\mathscr{S} \subset \mathscr{G}$ with card $\mathscr{S} < k+1$, then $\pi\mathscr{G} \neq \emptyset$. Helly's theorem asserts that $\alpha(\mathscr{C}^n) = n+1$ when $\mathscr{C}^n$ is the family of all convex subsets of R^n. The notion of Helly-number is especially appropriate for families $\mathscr{F}$ which are *intersectional* in the sense that $\pi\mathscr{G} \in \mathscr{F}$ for all $\mathscr{G} \subset \mathscr{F}$. For example, Grünbaum [10] defines a *measure of nonconvexity* $\Delta(X) \geqq 0$ for each compact $X \subset E^n$, with $\Delta(X) = 0$ characterizing the convex sets and $\Delta(X) < \infty$ if and only if X is the union of a finite family of pairwise disjoint compact convex sets. He proves

4.8. *For finite* $\varepsilon \geqq 0$, *let* $\mathscr{G}(n, \varepsilon)$ *denote the family of all compact sets* $X \subset E^n$ *with* $\Delta(X) \leqq \varepsilon$. *Then* $\mathscr{G}(n, \varepsilon)$ *is intersectional and has finite Helly-number.*

Of course $\alpha(\mathscr{G}(n, 0)) = n+1$. Grünbaum gives estimates for $\alpha(\mathscr{G}(n, \varepsilon))$ and examples showing the impossibility of improvement in certain directions.

The following result was proved by Motzkin [2]:

4.9. *Let* $\mathscr{M}(n, d)$ *denote the family of all varieties in (real) affine or projective* n*-space which are defined by one or more algebraic equations of degree* $\leqq d$. *Then* $\mathscr{M}(n, d)$ *is intersectional and* $\alpha(\mathscr{M}(n, d)) = \binom{n+d}{n}$.

In seeking Helly-type theorems for various non-intersectional families $\mathscr{F}$, one may encounter difficulties caused by the fact that the intersections of members of $\mathscr{F}$ can be structurally much more complicated than the individual members. For such $\mathscr{F}$, it may happen that the Helly-number $\alpha(\mathscr{F})$ is infinite and that a more appropriate notion is the *Helly-order* $\alpha^0(\mathscr{F})$, defined as the smallest cardinal k such that whenever $\mathscr{G}$ is a finite subfamily of $\mathscr{F}$ and $\emptyset \neq \pi\mathscr{S} \in \mathscr{F}$ for all $\mathscr{S} \subset \mathscr{G}$ with card $\mathscr{S} < k+1$, then $\emptyset \neq \pi\mathscr{G} \in \mathscr{F}$. Note that always $\alpha^0(\mathscr{F}) \leqq \alpha(\mathscr{F})$, with equality whenever $\mathscr{F} \cup \{\emptyset\}$ is intersectional.

Now let $\mathscr{U}(n, j)$ denote the family of all nonempty subsets of R^n which are the union of j or fewer disjoint compact convex sets. Then $\alpha(\mathscr{U}(n, 1)) = \alpha^0(\mathscr{U}(n, 1)) = n+1$ by Helly's theorem. An example of Motzkin (in the correction of Santaló [7]; see also §25 of Hadwiger-Debrunner [3]) shows that $\alpha(\mathscr{U}(n, 2)) = \aleph_0$. However, Grünbaum-Motzkin [1] establish

4.10. *The family of all "twins" in* E^n *has Helly-order* $2n+2$; *i.e.*, $\alpha^0(\mathscr{U}(n, 2)) = 2n+2$.

The same result holds if "compact" is replaced by "open" in the definition of $\mathscr{U}(n, j)$, but not if it is deleted. It is unknown even whether the family

$\mathscr{U}(2,3)$ of all "triplets" in R^2 is of finite Helly-order, but Grünbaum and Motzkin conjecture, more generally, that $\alpha^0(\mathscr{U}(n,j)) = j(n+1)$. Their paper contains other intersection theorems in an abstract setting, with interesting applications to convexity. In particular, they show that if $\mathscr{F}$ is a family of subsets of R^n, and the intersection of each n members of $\mathscr{F}$ is the union of n or fewer pairwise disjoint closed convex sets, then $\pi\mathscr{F}$ is itself such a union. (This is a theorem of Helly-type in the sense described below.)

A *nonempty* compact metric space is called a *homology cell* provided it is homologically trivial (acyclic) in all dimensions. In his second paper on intersection properties, Helly [2] employed Vietoris cycles and homology over the integers mod 2 to prove

4.11. *In R^n, the family of all homology cells has Helly-order $n+1$.*

A combinatorial form of this result appears on p. 297 of Alexandroff-Hopf [1]. Helly actually proved a somewhat stronger formulation: *If $\mathscr{F}$ is a family of subsets of $R^n, \pi\mathscr{G}$ is a homology cell for all $\mathscr{G} \subset \mathscr{F}$ with* card $\mathscr{G} \leqq n$, *and $\pi\mathscr{G} \neq \emptyset$ for all $\mathscr{G} \subset \mathscr{F}$ with* card $\mathscr{G} = n+1$, *then $\pi\mathscr{F}$ is a homology cell.*

It is interesting to contrast the fate of Helly's topological Theorem 4.11 with that of its special case concerning the intersection of convex sets. For the geometric theorem, many generalizations, variations, and applications were found, but the topological theorem was nearly forgotten during more than twenty years. Except in Helly's paper [2] and the book of Alexandroff-Hopf [1], we were unable to find any use of it prior to Molnár's paper [1] in 1956.

In R^2, Helly's topological theorem applies to simply connected compact sets. Helly [2] gave an elementary proof for this case, and Molnár [1; 2] established the following improvement: *A family of at least three simply connected compact sets in R^2 has nonempty simply connected intersection provided each two of its members have connected intersection and each three have nonempty intersection.*[2]

Now for a metric space X, let $\mathscr{H}_n X$ denote the family of all homology cells in X which can be topologically embedded in R^n. From Helly's result it follows easily that $\alpha^0(\mathscr{H}_n X) \leqq n+2$, whence of course $n+1 \leqq \alpha^0(\mathscr{H}_n M^n) \leqq n+2$ for every n-dimensional manifold M^n. As observed by Molnár [3], the 2-sphere S^2 is the only 2-dimensional manifold in which the family of homology cells is of Helly-order 4. More generally, it is evident that $\alpha^0(\mathscr{H}_n S^n) = n+2$, and another result of Helly [2] (also p. 296 of Alexandroff-Hopf [1]) implies that X contains a homology n-sphere whenever $\alpha^0(\mathscr{H}_n X) = n+2$. (See Soos [1] and Molnár [4] for related material.)

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[2] Using reduced Čech homology groups with coefficients in an arbitrary field, J. Berstein [1] has generalized Molnár's theorem as follows: *If $A_1, \cdots, A_m$ are compact subsets of an n-manifold M such that $A_1 \cup \cdots \cup A_m \neq M$ and such that for each $k \leqq n+1$ and for every $i_1, \cdots, i_k$ the set $A_{i_1} \cap \cdots \cap A_{i_k}$ is $(n-k)$-acyclic, then every union of intersections of the A_i is ∞-acyclic. In particular, the set $A_1 \cap \cdots \cap A_m$ is nonempty and ∞-acyclic.* (Here (-1)-acyclic means nonempty and 0-acyclic means connected.) See also Jussila [1].

There are many geometric problems in which the application of Helly's original intersection theorem (if possible at all) may require considerable ingenuity, but to which Helly's topological theorem can be applied quite simply. Apparently this was first noticed by Grünbaum [11] in connection with common transversals (see § 5). Undoubtedly, many new fields of applicability of Helly's topological theorem remain to be discovered.

The topological intersection theorem and other results of Helly [2] are closely related to the fact that *if a space be covered by a finite family* $\mathscr{F}$ *of sets all of whose intersections are homologically trivial, and all members of* $\mathscr{F}$ *are closed or all are open, then the space is homologically similar to the nerve* $N\mathscr{F}$. (For explicit formulation of one version of this result, see p. 138 of Leray [1] or § 4 of Chapter IV of Borel [1]. A related result on the Euler characteristic is given by Borsuk [3] and there are similar results involving homotopy type (Borsuk [2], Weil [1]).) As an illustration, let us derive Helly's geometric theorem from these results and the fact that every subset of R^n is homologically trivial in all dimensions $\geqq n$. Let $\mathscr{F}$ be a finite family of at least $n+1$ compact convex sets in R^n, each $n+1$ of which have a common point. Suppose $\pi\mathscr{F} = \emptyset$ and let k be the smallest number such that some $(k+1)$-membered subfamily $\mathscr{G}$ of $\mathscr{F}$ has empty intersection. Since each k members of $\mathscr{G}$ have a common point, the nerve $N\mathscr{G}$ is isomorphic with the $(k-1)$-skeleton S^{k-1} of a k-simplex. Since $N\mathscr{G}$ is homologically similar to the union U of all members of $\mathscr{G}$, it follows that the $(k-1)$-dimensional homology of U is not trivial. But this is impossible, for $U \subset R^n$ and $k-1 \geqq n$.

* * * * * * * * *

Helly's geometric theorem gives some information about "intersection patterns" of convex sets in R^n, and additional information is supplied by the result of Hadwiger-Debrunner [2] (4.4 above). It seems natural to ask what is meant by "an intersection pattern of convex sets in R^n" and, having settled on a definition, to attempt to determine all such patterns. We suggest two alternative but essentially equivalent definitions:

An intersection pattern of convex sets in R^n *is the*

$$\left\{\begin{matrix} \textit{nerve } N\mathscr{F} \textit{ of a finite family } \mathscr{F} \\ \textit{incidence array } A(F_1, \cdots, F_k) \textit{ of an ordered } k\textit{-tuple} \end{matrix}\right\} \textit{ of convex subsets of } R^n.$$

In the first case, $N\mathscr{F}$ is an abstract complex whose vertex-domain is $\mathscr{F}$, with $\mathscr{G} \in N\mathscr{F}$ (i.e., $\mathscr{G}$ is the vertex set of a simplex in $N\mathscr{F}$) if and only if $\pi\mathscr{G} \neq \emptyset$. The problem is to characterize intrinsically the abstract complexes which (up to isomorphism) can be obtained in this way. In the second case, $A(F_1, \cdots, F_k)$ is a function on $\{1, \cdots, k\}^k$ to $\{0, 1\}$ such that $A(i_1, \cdots, i_k) = 1$ if and only if $\bigcap_{\alpha=1}^{k} F_{i_\alpha} \neq \emptyset$. The problem is to characterize intrinsically the arrays which can be obtained in this way. Of course the two problems are equivalent, but the different formulations may suggest different approaches. It seems that the ideas of Hadwiger [2; 5] might be useful here.

Notice that the desired characterizations must depend on the dimension of the space, for with unrestricted dimension every intersection pattern is possible. Helly's theorem implies that for families in R^n the complex $N\mathscr{F}$ is determined

by its n-dimensional skeleton and the function $A(F_1, \cdots, F_k)$ is determined by its restriction to k-tuples $(i_1, \cdots, i_k)$ having at most $n+1$ different entries. Alternatively, one may consider in place of $A(F_1, \cdots, F_k)$ the analogously defined array $M_n(F_1, \cdots, F_k)$ which is a function on $\{1, \cdots, k\}^{n+1}$ to $\{0, 1\}$, and expresses only the incidence of j-tuples of the F_i's for $j \leqq n+1$. Further restrictions are imposed by the result of Hadwiger-Debrunner [2]. However, the problem of general characterization appears to be very difficult and is not trivial even for the case of R^1.[8]

The one-dimensional case was encountered by Seymour Benzer [1; 2] in connection with a problem in genetics. He was concerned with those $k \times k$ matrices which have the form $M_2(F_1, \cdots, F_k)$ for some k-tuple of intervals on the line. Alternatively, one may ask for an intrinsic description of those graphs G which are representable by intervals—that is, which can be realized as the 1-skeleton of a nerve $N\mathscr{F}$ for some finite family $\mathscr{F}$ of linear intervals. In this form, the problem was recently solved by Lekkerkerker-Boland [1], whose result is as follows:

4.12. *A finite graph G is representable by intervals if and only if it satisfies the following two conditions:*

(α) *G does not contain an irreducible cycle with more than three edges;*

(β) *among any three vertices of G, at least one is directly connected to each path joining the other two.*

They also characterize the representable graphs as those which fail to have subgraphs of certain types, and they describe some practical methods for deciding whether a given graph is representable by intervals.

We turn now to an abstract description of some of the types of problems which arise in connection with Helly's theorem and its relatives. Suppose X is a given space and $\mathfrak{P}$ is a hereditary property of families of subsets of X. That is, $\mathfrak{P}$ is a class of families of subsets of X, and $\mathscr{F} \subset \mathscr{G} \in \mathfrak{P}$ implies $\mathscr{F} \in \mathfrak{P}$. For each cardinal κ, define the property $\mathfrak{P}_\kappa$ by agreeing that $\mathscr{G} \in \mathfrak{P}_\kappa$ provided $\mathscr{F} \in \mathfrak{P}$ whenever $\mathscr{F} \subset \mathscr{G}$ and $\operatorname{card} \mathscr{F} < \kappa + 1$. (Use of the condition "$\operatorname{card} \mathscr{F} < \kappa + 1$" rather than "$\operatorname{card} \mathscr{F} < \kappa$" is motivated by the feeling of Grünbaum and Klee that the Helly-number $\alpha_{\aleph_0}(\mathscr{C}^n)$ defined below should turn out to be $n+1$ rather than $n+2$. If this requirement were abandoned, less awkward formulations would result from using "$\operatorname{card} \mathscr{F} < \kappa$," a course favored by Danzer. We use "$\operatorname{card} \mathscr{F} < \kappa + 1$" rather than "$\operatorname{card} \mathscr{F} \leqq \kappa$" in order to handle infinite as well as finite cardinals.)

A problem of *Helly-type* is to find conditions under which $\mathscr{F} \in \mathfrak{P}_\kappa$ implies $\mathscr{F} \in \mathfrak{P}_\lambda$, where κ and λ are cardinals with $\kappa < \lambda$. The conditions may be on the individual members of $\mathscr{F}$ as well as on the structure of $\mathscr{F}$, and of course will depend on $X, \mathfrak{P}, \kappa$ and λ. Essentially the same problem is often viewed the other way around: Given $X, \mathfrak{P}$, conditions on $\mathscr{F}$, and κ [resp. λ], what is the greatest λ [resp. the smallest κ] such that $\mathscr{F} \in \mathfrak{P}_\kappa$ implies $\mathscr{F} \in \mathfrak{P}_\lambda$?

[8] It seems difficult even to determine all intersection patterns which are realized by parallelotopes in R^n whose edges are parallel to the coordinate axes. Some special aspects of this problem have recently been treated by Danzer and Grünbaum.

A more general problem is said here to be of *Gallai-type*. Suppose $X, \mathfrak{P}, \kappa, \lambda$, and conditions on $\mathscr{F}$ are given. Even though $\mathscr{F} \in \mathfrak{P}_\kappa$ may not imply $\mathscr{F} \in \mathfrak{P}_\lambda$, it is clear that each $\mathscr{F} \in \mathfrak{P}_\kappa$ can be split into subfamilies having the property $\mathfrak{P}_\lambda$. The problem is to determine the smallest cardinal ψ such that every family $\mathscr{F}$ which satisfies the given conditions and has property $\mathfrak{P}_\kappa$ can be split into ψ (or fewer) subfamilies each having property $\mathfrak{P}_\lambda$.

See 8.8+ for a still more general type of problem.

Problems of these types have been considered in the literature mainly for convex subsets of linear spaces, and the remaining work is almost entirely for spherical spaces (see § 9).[4] Nevertheless, we believe that of the many theorems reported here which can be formulated without linearity, most have counterparts not only for the sphere but for any manifold. Of course the property of special interest is that of having nonempty intersection.[5] We denote this by $\mathfrak{D}$, so that $\mathscr{F} \in \mathfrak{D}$ if and only if $\pi\mathscr{F} \neq \emptyset$. The literature deals only with problems in which κ is given extremely small or λ is given extremely large. In fact $\mathfrak{P}_2$ is the simplest interesting property and in the other direction $\mathfrak{P}_{\aleph_0}$ (denoted henceforth by $\mathfrak{P}_\infty$) is of special interest since when $\mathscr{F} \in \mathfrak{P}_\infty$ a compactness argument can often be used to show that $\mathscr{F} \in \mathfrak{P}$. Note that $\mathscr{F} \in \mathfrak{D}_2$ provided each two members of $\mathscr{F}$ have a common point and $\mathscr{F} \in \mathfrak{D}_\infty$ provided $\mathscr{F}$ has the finite intersection property. In a sense $\mathfrak{D}_\infty$ marks the borderline between combinatorial geometry and general topology.

Existing results on the above problems involve principally the numbers $\alpha_\lambda(\mathscr{C}), \beta_\kappa(\mathscr{C})$, and $\gamma_\kappa(\mathscr{C})$ which are defined below in terms of the property $\mathfrak{D}$. Note, however, that the definitions may be given for other properties as well, and that some interesting problems can be formulated in this way. Let $\mathscr{C}$ be a family of sets. Then:

the λth *Helly-number* $\alpha_\lambda(\mathscr{C})$ is the smallest cardinal κ such that ($\mathscr{F} \subset \mathscr{C}$ and $\mathscr{F} \in \mathfrak{D}_\kappa$) implies $\mathscr{F} \in \mathfrak{D}_\lambda$;

the κth *Hanner-number* $\beta_\kappa(\mathscr{C})$ is the largest cardinal λ with $\lambda + 1 \leqq$ (successor of card $\mathscr{C}$), and such that ($\mathscr{F} \subset \mathscr{C}$ and $\mathscr{F} \in \mathfrak{D}_\kappa$) implies $\mathscr{F} \in \mathfrak{D}_\lambda$;

the κth *Gallai-number* $\gamma_\kappa(\mathscr{C})$ is the smallest cardinal μ such that each $\mathscr{F} \subset \mathscr{C}$ which is in $\mathfrak{D}_\kappa$ can be split into μ (or fewer) subfamilies, each having property $\mathfrak{D}_\infty$.

The Helly- and Hanner-numbers are dual in the sense that if card $\mathscr{C} \geqq \lambda$, then $\alpha_\lambda(\mathscr{C}) \leqq \kappa$ if and only if $\beta_\kappa(\mathscr{C}) \geqq \lambda$. Note also that $\alpha_{\aleph_0}(\mathscr{C}) = \alpha(\mathscr{C})$, where $\alpha(\mathscr{C})$ is the Helly-number defined earlier in this section, and that $\alpha_{\aleph_0}(\mathscr{C}) \leqq \delta$ iff $\gamma_\delta(\mathscr{C}) = 1$. Helly's theorem asserts that if $\mathscr{C}^n$ is the family of all convex subsets of R^n, then $\alpha_{\aleph_0}(\mathscr{C}^n) = n + 1$ and $\beta_{n+1}(\mathscr{C}^n) = \aleph_0$. § 7 discusses the Hanner- and Gallai-numbers for certain subclasses of $\mathscr{C}^n$ and describes the problems of Hanner [1] and Gallai (in Fejes Tóth [3]) which motivated this terminology.

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[4] From an abstract combinatorial viewpoint, such problems have recently been studied by Danzer.

[5] Also of special interest is the property $\mathfrak{Q}$ of being disjoint. That is, $\mathscr{F} \in \mathfrak{Q}$ iff $F_1 \cap F_2 = \emptyset$ whenever $F_i \in \mathscr{F}$ with $F_1 \neq F_2$. Cf. 8.6–8.8.

5. Common transversals. An *m-transversal* of a family of sets in R^n is an m-dimensional flat which intersects each member of the family. Thus Helly's theorem deals with 0-transversals and Horn's generalization 4.3 assures the existence of m-transversals under certain conditions on the existence of 0-transversals. The present section is devoted to the following problem, first formulated by Vincensini [1]: For $1 \leqq m \leqq n-1$ and $j \geqq 2$, find conditions on a family $\mathscr{F}$ of sets in R^n which assure that if each j-membered subfamily of $\mathscr{F}$ admits an m-transversal, then so does $\mathscr{F}$. At present, only a little is known about the case $m = n-1$ (see 5.1 and 5.2). All the other results are for $m = 1$, and some only for $n = 2$. The literature abounds in examples showing that even in R^2, rather stringent conditions must be imposed (Frucht [1], Grünbaum [1; 6], Hadwiger [9], Hadwiger-Debrunner [1; 3], Harrop-Rado [1], Kijne [1], Kuiper [1], and Santaló [1; 2]).

Vincesini's problem involves the manifold $F_{n,m}$ of all m-flats in R^n. While $F_{n,0}$ is of course identifiable with R^n, the manifolds $F_{n,m}$ are not contractible for $1 \leqq m \leqq n-1$, and herein lies part of the difficulty. It seems that any comprehensive study of m-transversals, seeking new theorems under minimal hypotheses, must take account of known results on the structure of $F_{n,m}$ and of the closely associated Grassmannian manifold $G_{n+1,m+1}$ of all $(m+1)$-dimensional linear subspaces of R^{n+1}. For accounts of some such results and references to others, see Steenrod [1] and Milnor [1].

The first significant results on common transversals were those of Santaló [1], who proved

5.1. *If $\mathscr{P}$ is a family of parallelotopes in R^n with edges parallel to the coördinate axes, and an $(n-1)$-transversal is admitted by each subfamily $\mathscr{Q} \subset \mathscr{P}$ with* card $\mathscr{Q} \leqq 2^{n-1}(n+1)$, *then $\mathscr{P}$ itself admits an $(n-1)$-transversal.*

Santaló's proof is based on an analogue of Radon's theorem. By extending the methods of Hadwiger-Debrunner [1; 3] and Grünbaum [5; 6; 11], Grünbaum [20] has recently used Helly's theorem to obtain a generalization of 5.1. The following terminology is convenient: A convex cone C is an *associated cone* of a polyhedron P provided the vertex v of C is one of the vertices of P, and C is the cone from v generated by P (i.e., $C := \bigcup_{x \in P} v + [0, \infty[(x - v)$). A polyhedron P' is *related to* P provided each associated cone of P' is an intersection of translates of associated cones of P. Then Grünbaum's theorem is

5.2. *Suppose P is a centrally symmetric polyhedron in R^n with $2p$ vertices, and $\mathscr{P}$ is a family of polyhedra related to P. If an $(n-1)$-transversal is admitted by each subfamily $\mathscr{Q} \subset \mathscr{P}$ with* card $\mathscr{Q} \leqq p(n+1)$, *then $\mathscr{P}$ itself admits an $(n-1)$-transversal.*

Increasing the number $p(n+1)$ leads to a similar result without the assumption of central symmetry. Even under additional restrictions on the family $\mathscr{P}$, the bound in 5.1 is the best possible for $n = 2$. (See Frucht [1], Santaló [2], and Grünbaum [6].) The best bounds in higher dimensions are unknown.

For other results on transversal hyperplanes, see the paper of Valentine [9] in this volume.

The rest of the section will be devoted to 1-transversals. We say that a family $\mathscr{F}$ of sets has property $\mathfrak{T}$ provided $\mathscr{F}$ admits a 1-transversal, and property $\mathfrak{T}_j$ provided a 1-transversal is admitted by every subfamily consisting of at most j members of $\mathscr{F}$. The following result (Grünbaum [20]) is a partial converse of 5.2:

5.3. *If K is a centrally symmetric convex body in R^2 and if there exists $j < \infty$ such that $\mathfrak{T}_j$ implies $\mathfrak{T}$ for all families of homothets of K, then K is a polygon.*

An early result of Santaló [1] is

5.4. *For a family of parallelotopes in R^n with edges parallel to the coördinate axes, $\mathfrak{T}_{(2n-1)2^n}$ implies $\mathfrak{T}$.*

It would be interesting to find an extension (analogous to 5.2) of 5.4 to more general classes of polyhedra.

A family $\mathscr{K}$ of sets in R^n is said to be *separated* if there exists a finite sequence $\mathscr{H} = (H_0, \cdots, H_m)$ of parallel hyperplanes in R^n and an enumeration $(K_1, \cdots, K_m)$ of $\mathscr{K}$ such that each K_i lies in the open set bounded by H_{i-1} and H_i.

As a direct application of Helly's topological theorem, Grünbaum [11] proved

5.5. *Suppose $\mathscr{K}$ and $\mathscr{H}$ are as above, and for $1 \leqq i \leqq m$ let K_i^* be the set of all points $(u, v) \in H_0 \times H_m$ such that $[u, v]$ intersects K_i. If the sets K_i are all compact and convex and if the intersection of any $3, 4, \cdots, 2n-2$ of the sets K_i^* is a homology cell, then (for $\mathscr{K}$) $\mathfrak{T}_{2n-1}$ implies $\mathfrak{T}$.*

From this he deduced

5.6. *For a family of compact convex sets in R^n whose members lie in distinct parallel hyperplanes, $\mathfrak{T}_{2n-1}$ implies $\mathfrak{T}$.*

5.7. *For a family of Euclidean cells in E^n such that the distance between any two centers is greater than the sum of the corresponding diameters, $\mathfrak{T}_{2n-1}$ implies $\mathfrak{T}$.*

The result 5.6 is an easy consequence of Helly's geometric theorem, and was first proved for $n = 2$ by Santaló [2] (see also Dresher [1], Rademacher-Schoenberg [1]); 5.7 extends earlier results of Hadwiger [7] and Hadwiger-Debrunner [3].

With $\mathscr{K}$ and $\mathscr{H}$ as above, and $j \geqq 2$, $\mathscr{K}$ is called *j-simple* provided each j of the sets K_i^* have connected intersection. From reasoning similar to that of Brunn [2], it follows that every separated family of convex sets in R^n is 3-simple. With the aid of this remark, Grünbaum [11] derived the following from 5.5:

5.8. *For a 4-simple separated family of compact convex sets in R^3, $\mathfrak{T}_5$ implies $\mathfrak{T}$.*

His examples show that none of these conditions can be dropped. It would

be interesting to extend 5.8 to higher dimensions, and in particular to decide whether connectedness of sets of the form $\bigcap_t K_i^*$ implies that they are homology cells.

There are some theorems on common transversals for infinite families of sets which seem to have no counterpart for finite families. The following results from Hadwiger's proof [8] of a weaker statement:

5.9. *For a family of compact convex sets in E^n whose union is unbounded while the diameters of its members have finite upper bound, $\mathfrak{T}_{n+1}$ implies $\mathfrak{T}$.*

We turn finally to R^2, for which Vincensini's problem has been more thoroughly studied. Two approaches have led to interesting results. One permits the individual convex sets to be quite general, but places stringent restrictions on their relative positions in the plane. Generalizing earlier results of Vincensini [5], Klee [3], and Grünbaum [1], Hadwiger [10] proved the following (where a *disjoint family* of sets is one whose members are pairwise disjoint):

5.10. *A disjoint family $\mathscr{K}$ of compact convex sets in R^2 admits a 1-transversal if and only if $\mathscr{K}$ can be linearly ordered in such a way that each 3-membered subfamily of $\mathscr{K}$ admits a 1-transversal intersecting its members in the specified order.*

For $j \geqq 2$, a family $\mathscr{K}$ of sets in R^n will be said to have property $\mathfrak{S}_j$ provided each at most j-membered subfamily of $\mathscr{K}$ can be ordered in such a way that, for each i with $1 \leqq i \leqq j-1$, the convex hull of the first i members is disjoint from that of the remaining $j-i$ members. The main result of Kuiper [1] and Harrop-Rado [1] may be formulated as follows:

5.11. *A family of compact convex sets in R^2 has property $\mathfrak{T}$ if it has $\mathfrak{S}_4$ and $\mathfrak{T}_3$, and also if it has $\mathfrak{S}_3$ and $\mathfrak{T}_4$.*

The result was given in this form by Harrop-Rado. Kuiper's version was in the projective plane, which in a sense is the appropriate medium. Grünbaum [11] employed Helly's topological theorem to establish (in the projective plane) a stronger form of the second half of 5.11.

Another approach permits more freedom in the relative positions of the sets, but assumes that they are mutually disjoint and congruent. (The virtual necessity of these assumptions is shown by various examples (Hadwiger-Debrunner [1; 3]).)

5.12. *For a disjoint family of congruent circular discs in E^2, $\mathfrak{T}_5$ implies $\mathfrak{T}$. For such families with at least six members, $\mathfrak{T}_4$ implies $\mathfrak{T}$.*

5.13. *For a disjoint family of translates of a parallelogram in R^2, $\mathfrak{T}_5$ implies $\mathfrak{T}$.*

The first part of 5.12 is due to Danzer [1], solving a problem of Hadwiger-Debrunner [1] and Hadwiger [6]; 5.13 and the second part of 5.12 are due to Grünbaum [6]. For additional results on common transversals in the plane, see Grünbaum [11] and especially Hadwiger-Debrunner [3].

The following are conjectures of Grünbaum [6]. (1) $\mathfrak{T}_5$ implies $\mathfrak{T}$ for disjoint families of congruent squares in E^2, and for disjoint families of translates of an arbitrary convex body in R^2. (2) $\mathfrak{T}_6$ implies $\mathfrak{T}$ for disjoint families of congruent compact convex sets in E^2. The second conjecture is unsettled even for disjoint families of congruent segments.

We end the section by stating a result of Gallai type on common transversals, due to Hadwiger-Debrunner [1; 3]:

5.14. *For a family $\mathscr{K}$ of positive homothets of a compact convex set in R^2, $\mathfrak{T}_4$ implies that $\mathscr{K}$ can be split into 4 or fewer subfamilies, each having property $\mathfrak{T}$.*

6. Some covering problems. This section is included because Helly's theorem is a valuable tool in connection with some of the problems, and because certain covering theorems are used in § 7 to treat intersection properties of special families. Discussing only the results most directly related to Helly's theorem would yield a distorted picture of an interesting and active area of research, so we include additional material to round out the picture.

Our starting point is the theorem of Jung [1], whose popularity may be judged from its appearance in books by Bonnesen-Fenchel [1], Yaglom-Boltyanskiĭ [1], Hadwiger [15], Eggleston [3], and in many research papers. (To those cited by Blumenthal-Wahlin [1] and Hadwiger [15], we add Straszewicz [1], Lagrange [1], Verblunsky [1], Gustin [1], Ehrhart [1] and Melzak [1].) Jung's theorem, proved in 2.6 with the aid of Helly's theorem, asserts that each subset of E^n of diameter $\leqq d$ lies in a spherical cell of radius $\leqq [n/(2n+2)]^{1/2}d$, where of course the terms are defined with respect to the Euclidean distance. An interesting problem arises when this is replaced by other distance functions. It is discussed directly below, and other generalizations or relatives of the problem treated by Jung are discussed later in the section. Though the known results are rather scanty in each case, general formulations will suggest avenues for further study.

* * * * * * * * *

Suppose E is a set and ρ is a function on $E \times E$ to $[0, \infty[$. For each point $x \in E$ and each $\varepsilon > 0$, let $V_\rho(x, \varepsilon) := \{y \in E : \rho(x, y) \leqq \varepsilon\}$. For $x \in E$ and $Y \subset E$, the *(ρ, x)-radius* of Y is the number

$$r_{\rho,x}(Y) := \sup \{\rho(x, y) : y \in Y\} = \inf \{\varepsilon \geqq 0 : Y \subset V_\rho(x, \varepsilon)\} ,$$

and the *ρ-radius* of Y is the number

$$\mathrm{rad}_\rho\, Y := \inf \{\varepsilon \geqq 0 : \exists x \in E \quad \text{with} \quad Y \subset V_\rho(x, \varepsilon)\} = \inf \{r_{\rho,x}(Y) : x \in E\} .$$

The *ρ-diameter* of $Y \subset E$ is the number

$$\mathrm{diam}_\rho\, Y = \sup \{\rho(y, y') : y, y' \in Y\} = \sup \{r_{\rho,y}(Y) : y \in Y\} .$$

Now for two (possibly different) functions ρ and ρ' on $E \times E$ to $[0, \infty[$, the problem is to determine the function $J(\rho, \rho'; d)$ defined as follows for each $d \geqq 0$:

$$J(\rho, \rho'; d) = \sup\{\mathrm{rad}_\rho Y : Y \subset E, \mathrm{diam}_{\rho'} Y \leqq d\} .$$

Existing literature treats only very special cases of the problem, and in particular is restricted to the case $\rho = \rho'$. Accordingly, we define

$$J(\rho, d) := J(\rho, \rho; d) .$$

In this notation, Jung's theorem asserts that if ρ_0 is the (Euclidean) distance in E^n, then $J(\rho_0, d) = [n/(2n+2)]^{1/2}d$. Santaló [5] describes the function $J(\rho, d)$ where ρ is geodesic distance on the unit sphere S^n in E^{n+1}. (See also the discussion of spherical convexity in 9.1.)

The above formulation subsumes various problems concerning covering by positive homothets of a set in a linear space E. In particular, suppose C is a convex body in E with $0 \in \mathrm{int}\, C$, and for $x, y \in E$ let

$$\rho_C(x, y) := \inf\{t \geqq 0 : y - x \in tC\} .$$

Then ρ satisfies the triangle inequality and is symmetric when $C = -C$. (For metric spaces with nonsymmetric distance, see Zaustinsky [1].) The function $J(\rho_C, d)$ is positively homogeneous in d and hence is determined by the value of $J(\rho_C, 1)$. It is known that $J(\rho_C, 1) \leqq n(n+1)^{-1}$ when E is n-dimensional and $C = -C$ (see 6.4-6.5 below).

In the above setting, it is natural to discuss centers as well as radii of sets, where a *ρ-center* of Y is a point $x \in E$ for which $r_{\rho,x}(Y) = \mathrm{rad}_\rho Y$. For distances ρ_C, such centers always exist in the finite-dimensional case, and under the Euclidean metric the center is unique (for bounded Y) and lies in the closed convex hull of Y. The last condition is almost characteristic of the Euclidean metric, for Klee [5] has proved the following:

6.1 *For a normed linear space E, the following three assertions are equivalent:*
a^0 *E is an inner-product space or is two-dimensional;*
b^0 *if a subset Y of E lies in a cell of radius <1, then Y lies in some cell of unit radius centered at a point of* conv Y;
c^0 *if a subset Z of E lies in a cell of radius <1, then Z is intersected by every cell of unit radius centered at a point of* conv Z.

Results related to the equivalence $a^0 \Leftrightarrow b^0$ were also obtained by Kakutani [1], Grünbaum [7] and Comfort-Gordon [1], the first and last making use of Helly's theorem.

* * * * * * * * *

A basic relationship between covering and intersection properties was suggested by 2.1. In particular, the following simple corollary of Helly's theorem is very useful:

6.2. *If $X \subset R^n$ and each $n+1$ or fewer points of X can be covered by some translate of the convex body $K \subset R^n$, then X lies in some translate of K.*

Closer inspection of the reasoning for 6.2 leads to the following lemma, trivial but fundamental.

6.3. *Suppose G is a group (written additively but not assumed to be abelian), W and Z are subsets of G, and κ and λ are cardinal numbers. Then the following two statements are equivalent:*

a° *whenever $Y \subset G$ with* card $Y < \lambda + 1$, *and every set $X \subset Y$ with* card $X < \kappa + 1$ *lies in some member of $\{g + W : g \in G\}$, then Y lies in some member of $\{g + Z : g \in G\}$;*

b° *whenever $U \subset G$ and the family $\{W + u : u \in U\}$ is such that each of its subfamilies of cardinality $< \kappa + 1$ has nonempty intersection, then each subfamily of $\{Z + u : u \in U\}$ of cardinality $< \lambda + 1$ has nonempty intersection.*

PROOF. Suppose a° holds and consider U as in b°. Let $V \subset U$ with card $V < \lambda + 1$. We wish to show that $\bigcap_{v \in V} (Z + v)$ is nonempty, or, equivalently, that $-V$ lies in some translate of Z. But this follows from a°, for the hypothesis of b° insures that every subset of $-V$ of cardinality $< \kappa + 1$ lies in some translate of W. Thus a° implies b°, and it follows similarly that b° implies a°.

It is also easy to prove a generalization of 6.3 which is related to 6.3 as 2.1 is to 6.2. From 6.3 it follows that with W and λ fixed and $Z = W$, the Helly-number $\alpha_\lambda(\{W + g : g \in G\})$ (defined in § 4) is exactly the smallest cardinal κ for which a° holds. If every $\alpha(\{W + g : g \in G\})$ points of a finite subset Y of G can be covered by some left-translate of W, then Y itself can be so covered; under suitable compactness conditions, this applies to infinite sets Y as well.

The following is easily verified:

6.4. *For a convex body C in R^n with $0 \in \operatorname{int} C$, let J_C^* $[J_C]$ denote the smallest number $\sigma \geqq 0$ for which σC covers by translation every set Y such that $y' \in y + 2C$ for all $y, y' \in Y$ [each two pointed subset of Y lies in some translate of C]. Then always $J_C^* = J(\rho_C, 2)$, while $J_C^* = J_C$ when $C = -C$.*

The number J_C is called the *Jung constant* of C, and each translate of the set $J_C C$ is a *Jung cell* associated with C. Note that J_C^* depends on the position of C relative to the origin, while J_C is translation-invariant; also, diam $_{\rho_C} Y =$ diam $_{\rho_{C \cap (-C)}} Y$ and hence $J_C^* \leqq J_{C \cap (-C)}$. When $C = -C$, both the above conditions on Y express the fact that diam $_{\rho_C} Y \leqq 2$. In general, however, the two conditions are not equivalent and it is convenient to define a *C-distance* which is invariant under translation of C. This is merely the distance generated by the gauge functional of the Minkowski symmetrization C^* of C, where $C^* := (C + (-C))/2$. Equivalently,

$\| x - y \|_C := \inf \{2\alpha \geqq 0 : \{x, y\}$ lies in some translate of $\alpha C\}$, and $\operatorname{diam}_C Y =$ diam $_{\rho_{C^*}} Y = \sup \{\| x - y \|_C : x, y \in Y\}$.

The following is known:

6.5. *If C is a convex body in R^n with $0 \in \operatorname{int} C$, then $J_C \leqq n$ and $J_C^* \leqq 2n(n+1)^{-1}$ (whence $J_C \leqq 2n(n+1)^{-1}$ when $C = -C$).*

The inequality on J_C^* was proved by Bohnenblust [1], and later in simpler

fashion by Leichtweiss [1] and Eggleston [2]. The same result (in the language of analysis) is hidden in Volkov [1]. Leichtweiss also described the bodies for which the maximum value of J_C^* is attained. A body $C \subset R^n$ will here be called a *Leichtweiss body* provided there is a simplex T such that $T + (-T) \subset 2C \subset (n+1)(-T)$ (which imples that the origin is the centroid of T). Leichtweiss [1] showed that for a convex body $C \subset R^n$, $J_C^* = 2n(n+1)^{-1}$ if and only if C is a Leichtweiss body; the convex hull of the corresponding Y is necessarily (up to translation) such a T. With this result one proves easily that $J_C \leqq n$ with equality exactly when C is a simplex; the corresponding set conv Y is a translate of $-C$.

The above results were obtained also by Grünbaum [2; 8], who employed the following definitions:

6.6. *For a convex body $C \subset R^n$ with $0 \in \operatorname{int} C$, the Jung constant J_C [expansion constant E_C] is the smallest number $\sigma > 0$ such that whenever $\{x + C : x \in X\}$ $[\{x + \alpha_x C : x \in X\}]$ is a family of pairwise intersecting translates [(positive) homothets] of C, then the expanded family $\{x + \sigma C : x \in X\}[\{x + \sigma\alpha_x C : x \in X\}]$ has nonempty intersection.*

From 6.3 it is clear that this definition of J_C agrees with the earlier one. In particular, J_C is translation-invariant though E_C is not. Clearly $1 \leqq J_C \leqq E_C$, and from results of Nachbin [1], Sz.-Nagy [1] and Hanner [1], it follows that $J_C = 1$ or $E_C = 1$ exactly when C is a parallelotope (cf. 7.1). Grünbaum [8] proved

6.7. *For a convex body C in R^n with $C = -C$, $1 \leqq J_C \leqq E_C \leqq 2n(n+1)^{-1}$. Even in the plane, J_C and E_C need not be equal. However, they are equal whenever either is equal to 1 or to $2n/(n+1)$, and also when C is an ellipsoid (a Euclidean cell).*

For the close relationship between expansion constants and the extension of linear transformations, see 7.2.

Now for a cardinal $j \geqq 2$, let J_C^j and E_C^j be defined as in 6.6 with respect to families of cells having the property $\mathfrak{D}_j$ rather than merely $\mathfrak{D}_2$. Thus, for example, $J_C^2 = J_C$ and J_C^3 is defined by substituting "triply" for "pairwise" in 6.6. By Helly's theorem $J_C^j = E_C^j = 1$ for every convex body $C \subset R^n$ and $j \geqq n+1$. The following is known:

6.8. *If B^n is the n-dimensional Euclidean cell, then*

$$E_{B^n}^j = J_{B^n}^j = [nj(n+1)^{-1}(j-1)^{-1}]^{1/2}$$

for $2 \leqq j \leqq n+1$.

The second equality is due to Danzer (unpublished manuscript; see also a comment in the proof of 2.6) and the first follows from reasoning employed by Grünbaum [8] for the case $j = 2$.

It is interesting to note that

$$\max\{J_C^3 : C = -C, C \text{ a convex body in } R^3\} = 3/2\ ,$$

which is also the maximum of J_C^2. (If $C = -C$ and $J_C^3 = 3/2$, then of course $J_C^2 = 3/2$ and C is a Leichtweiss body. On the other hand, every Leichtweiss body $D \subset R^3$ which is also a *Hanner body* (see 7.1 ff.) has $J_D^3 = 3/2$, and the regular octahedron is such a body.)

See 7.5–7.6 for the relationship between Gallai's problem and the numbers J_C^j.

Another generalization of Jung's problem is the following: Given a class $\mathscr{Y}$ of subsets of a linear space E and a class G of transformations of E into E, a set $U \subset E$ is called a *G-universal cover* of $\mathscr{Y}$ provided for each $Y \in \mathscr{Y}$ there exists $g \in G$ such that $gU \supset Y$. For a set C which is starshaped from the origin in E, the problem is to evaluate the number

$$\inf \{\sigma \geqq 0 : \sigma C \text{ is a } G\text{-universal cover of } \mathscr{Y}\} .$$

In addition to the results on Jung constant, the following is known:

6.9. *Suppose C is a centrally symmetric convex body in R^n, T is a simplex containing C, and Y is a subset of R^n such that* $\operatorname{diam}_C Y \leqq 2$ (*i.e., each two-pointed subset of Y lies in some translate of C*). *Then Y can be covered by a translate of T or by a translate of $-T$.*

Proof. Let T_1 be the smallest homothet (positive or negative) of T which covers Y and let $x + \alpha C$ be the largest positive homothet of C which lies in T_1. We claim that Y is covered by the simplex $T_2 := (2 - \alpha)\alpha^{-1}(2x + (-T_1))$. If this be so, then of course $\alpha \leqq 1$ (for otherwise T_2 is smaller than T_1) and the proof is complete.

To see that $T_2 \supset Y$, note that T_2 is a negative homothet of T_1 such that corresponding faces of dimension $n-1$ lie in parallel hyperplanes at C-distance 2. Then use the facts that $\operatorname{diam}_C Y \leqq 2$ and $\operatorname{cl} Y$ intersects every $(n-1)$-dimensional face of T_1.

The following can be derived from 6.9 by standard continuity arguments.

6.10. *Suppose G is the group of (orientation-preserving) motions in E^n, S is an n-simplex circumscribed to the Euclidean unit cell such that $-S \in GS$, and $\mathscr{Y}$ is the class of all sets $Y \subset E^n$ of Euclidean diameter $\leqq 2$. Then $S \cap -S$ is a G-universal cover of $\mathscr{Y}$.*

Another consequence of 6.9 is that if C is a centrally symmetric convex body in R^n and T is a simplex of least volume among those which will cover (after suitable translation or point reflection) every set of C-diameter 2, then T is a translate of a simplex of least volume among those which are circumscribed to C.

The special case of 6.10 where S is regular is due to Gale [2]. The above proof of 6.9 is essentially due to Viet [1], a student of Süss. Using the same idea, Süss himself [1] gave an elegant proof of Gale's theorem and tried to employ this for a short proof of Jung's theorem. Unfortunately, his calculation to this end contained an irreparable error, for if S is regular and circumscribed to B^n, then for $n \geqq 3$ the set $S \cap -S$ will not fit into the Jung cell $(J_{B^n})B^n$.

Theorem 6.9 remains true when the assumptions are replaced by: "Suppose C is a convex body in R^n, S is a simplex containing $(C + (-C))/2 \cdots$." It

would be interesting to find a similar result under the original assumption on S but dropping the symmetry of C.

* * * * * * * * *

Instead of asking *how large* σ must be for σC to cover any set of C-diameter 2, one may ask, for fixed σ, *how many* translates of σC are needed. In particular, $\delta(C)$ will denote the smallest integer k such that every set of C-diameter 2 can be convered by k translates of C.

Grünbaum [3] has proved

6.11. *If C is a centrally symmetrix convex body in R^2, the Jung cell $J_C C$ can be covered by three translates of C; hence $\delta(C) \leqq 3$.*

For related results on the Euclidean plane, see Gale [2], Grünbaum [17], and papers by H. Lenz listed by Grünbaum [17]. Grünbaum conjectures that $\delta(B^n) = n + 1$, even though for $n \geqq 3$ it is clear that the Jung cell $(J_{B^n})B^n$ cannot be covered by $n + 1$ translates of B^n. The best known result in the direction of this conjecture is the following, due to Danzer (unpublished manuscript).

6.12. *The Jung cell $\sqrt{2n(n+1)^{-1}}B^n$ can be covered by*

$$\sqrt{\frac{(n+2)^3}{3}}\left(\sqrt{2+\sqrt{2}}\right)^{n-1}$$

translates of B^n.[6]

Close numerical investigations starting from the universal cover of Grünbaum [4] and Heppes [1] seem to verify that $\delta(B^3) = 4$. For the relationship between $\delta(C)$ and Gallai's problem, see 7.5.

Grünbaum's conjecture and many other interesting questions are closely related to the famous conjecture of Borsuk [1] that in E^n, every set of diameter 1 can be split into $n + 1$ sets of smaller diameter (cf. Hadwiger [1; 3]). For the literature on Borsuk's conjecture and related matters, see the extensive report by Grünbaum [17] in this volume.

Another problem closely related to Borsuk's conjecture is due to Levi [3] (discussed also in two forms by Hadwiger [14; 16]): Given a convex body C in R^n, what is the smallest number $\varepsilon(C)$ of translates of $\operatorname{int} C$ to cover C. Obviously $\varepsilon(I^n) = 2^n$ for an n-dimensional parallelotope I^n, and probably $\varepsilon(C) \leqq 2^n$ for every convex body $C \subset R^n$. On the other hand, considering the spherical image of C obtained from parallel supporting hyperplanes, we see from the result of Borsuk [1] that $\varepsilon(C) \geqq n + 1$, with equality when C is smooth since the $(n-1)$-sphere can be covered by $n + 1$ open hemispheres. (This was the argument by which Hadwiger [1] proved Borsuk's conjecture for smooth bodies.) Levi [3] showed that $\varepsilon(C) \leqq 4$ in R^2, with equality characterizing the parallelograms (Levi [4]). See also Gohberg-Markus [1], and, for related problems, Boltyanskii [1] and Soltan [1].

[6] Perhaps this bound is already outdated by the method of Erdös-Rogers [1]. (Cf. the footnote to 7.10.)

Replacing the word "translates" in the definition of $\varepsilon(C)$, by "images under affine transformations, each of determinant one," leads to another constant $\varepsilon^*(C)$. Though easily seen that $\varepsilon^*(C) = 2$ in R^2 (as conjectured by Levi [3]), it is unknown whether $\varepsilon^*(C)$ assumes greater values in higher dimensions.

* * * * * * * * *

The covering theorems to be discussed next involve the group of isometries (Euclidean motions) in E^n and the group of affine transformations with determinant 1. Since both destroy any metrical concepts except those for the Euclidean distance, there is no longer any connection with Minkowskian geometry. Further, there is no analogue to 6.3 replacing translates by more general images. Nevertheless, the results 6.14–6.15 complete our knowledge of facts discussed earlier, and 6.17–6.18 are Helly-type theorems in the sense of § 4.

Considering circular discs D inscribed in a plane continuum C, Robinson [1] showed that if C itself is not a circular disc, there exists a positive ε such that any three points of the set $F = (1 + \varepsilon)D$ can be carried into C by a Euclidean motion. (He assumed superfluously that C was simply connected.) This proved

6.13. *In E^2 the circular discs are the only continua C with nonempty interior which have the following property:*

($\mathfrak{A}_3$) *If $F \subset E^2$ and each triple of points of F can be covered by some isometric image of C, then so can F itself.*

Soon afterwards, Santaló [3] proved the polar counterpart:

6.14. *In E^2 the circular discs are the only convex bodies C which have the following property:*

($\mathfrak{B}_3$) *If F is a convex body in E^2 and every triangle containing F contains also an isometric image of C, then F itself contains such an image.*

That the circular discs have properties $\mathfrak{A}_3$ and $\mathfrak{B}_3$ is evident from Helly's theorem (cf. 2.1). Blumenthal [1], Santaló [3], and Kelly [1] ask whether there are analogous results for other classes of plane convex bodies. In particular, what happens when the number *three* (of points in F for $\mathfrak{A}_3$, of sides of the polygon containing F for $\mathfrak{B}_3$) is replaced by some $k > 3$? Of course the circular discs still have the corresponding properties $\mathfrak{A}_k$ and $\mathfrak{B}_k$, but it is undecided whether they are still *characterised* by these properties. For $\mathfrak{A}_k$, the convex hull of a circular disc and a nearby point may be a counterexample; for $\mathfrak{B}_k$, intersect the disc with a halfplane.

Considering, for F, an insphere plus a thin "cap" of slightly larger radius (instead of a slightly expanded insphere), Danzer recently generalized 6.13 to E^3, assuming C to be a convex body. The corresponding polar argument is valid for 6.14. (Similar results have been reported by K. Böröczky in Budapest.)

Theorem 6.13 was inspired by the notion of the *congruence order* of a metric space, due to Menger [1] and studied extensively by Blumenthal [1; 3] and

others. A metric space M has *congruence indices* (n, k) with respect to a class $\mathscr{S}$ of metric spaces provided each member S of $\mathscr{S}$ with card $S > n + k$ can be isometrically embedded in M whenever such an embedding is possible for every $X \subset S$ with card $X \leqq n$. And M has *congruence order* n with respect to $\mathscr{S}$ provided $(n, 0)$ are congruence indices. (Note the close similarity of these ideas to those associated with Helly's theorem.) It is known that both E^n (in the Euclidean distance) and the Euclidean n-dimensional sphere S^n (in the geodesic distance) have congruence order $n + 3$ with respect to the class of all metric spaces (see §§ 38–39 of Blumenthal [3]). Among the plane convex bodies, the circular discs are characterized in 6.13 as having congruence order 3 with respect to the class $\mathscr{P}$ of all plane sets. Kelly [1] notes that except for the circular ones, no elliptical disc has finite congruence order. It would be interesting to characterize the plane convex bodies of congruence order k for $k > 3$.

A more algebraic problem is to determine the congruence indices of various curves in E^2. Though solved in a few cases (see Kelly [1], Blumenthal [3]), the problem is not settled even for all conics in E^2. It is known that each conic has congruence indices $(6, 0)$ and $(5, 1)$, but not whether it has $(5, 0)$. This asks the following: *Suppose Q is a conic in E^2, X is a set of six points in E^2, and for each $x \in X$ the set $X \sim \{x\}$ can be moved isometrically into Q. Must Q contain an isometric image of X?* (The problem is trivial when Q is a parabola. For Q an ellipse, an affirmative solution was recently obtained by J. J. Seidel and J. van Vollenhoven in Eindhoven.)

* * * * * * * * *

The affine transformations of determinant 1 carry ellipsoids in E^n onto other ellipsoids of equal volume. A basic fact is

6.15. *For each convex body C in R^n there is a unique circumscribed ellipsoid of least volume and a unique inscribed ellipsoid of greatest volume.*

Existence of the ellipsoids is trivial, and uniqueness of the first became well known since K. Löwner used it in his lectures; uniqueness of the second was apparently discovered independently by Danzer and Zaguskin. See Danzer-Laugwitz-Lenz [1] and Zaguskin [1] for proofs and applications.

Now let us say that a convex body C has property $\mathfrak{L}$ if its *Löwner-ellipsoid* has volume $\leqq 1$, *property* $\mathfrak{L}_k$ if conv X has property $\mathfrak{L}$ for each $X \subset C$ with card $X < k$, *property* $\mathfrak{J}$ if its *inellipsoid* has volume $\geqq 1$, and *property* $\mathfrak{J}_k$ if each intersection of fewer than k halfspaces, each containing C, has property $\mathfrak{J}$. With these definitions, we have

6.16. *If C is a convex body in R^n, $C \in \mathfrak{L}_{(n+1)(n+2)/2}$ implies $C \in \mathfrak{L}$.*

6.17. *If C is a convex body in R^2, $C \in \mathfrak{J}_6$ implies $C \in \mathfrak{J}$.*

For $n = 2$ these results are due to Behrend [1; 2], who also gave an exact description of how 3, 4, or 5 points on a circle [resp. tangents to a circle] must be located in order for the disc to be the Löwner-ellipsoid [resp. the

inellipsoid] of the polygon determined by these points [resp. by the tangents through these points]. The general result 6.16 was found by John [1], whose use of Lagrange's parameters should also be applicable to extend 6.17 to higher dimensions.

* * * * * * * * *

We turn now to concepts and results which are in a sense polar to some of the material treated earlier. This *polarity*, which appeared earlier between results of Robinson (6.13) and Santaló (6.14), involves a correspondence between points on the one hand and hyperplanes or halfspaces on the other. Though the correspondence is often called a *duality*, we prefer the weaker term since the relationship is much less strict than the true duality of, e.g., projective geometry. There is no exact *duality principle* but only a strong *duality feeling*. This leads to interesting new results and problems, but there are many true statements whose polar statements are false, and often it is even unclear how to "dualize" a given problem.

There seems to be no polar counterpart to 6.3, but the statement polar to 6.2 is the following consequence of 2.1:

6.18. *If a family $\mathscr{K}$ of closed halfspaces in R^n has bounded intersection, and each intersection of $n+1$ members contains some translate of the convex body C, then $\pi\mathscr{K}$ contains some translate of C.*

Now consider a convex body C in R^n and the associated C-distance $\|\ \|_C$ as defined in 6.4+. The C-distance between two parallel hyperplanes H and H' is $\inf\{\|x - x'\|_C : x \in H, x' \in H'\}$, or equivalently,

$$\|H, H'\|_C := \sup\{2\alpha \geqq 0\text{: some translate of } \alpha C \text{ fits between } H \text{ and } H'\}\ .$$

Further, the *C-width* of a bounded set $F \subset E^n$ is given by

$$\text{width}_C F = \inf\{\|H, H'\|_C : F \text{ lies between } H \text{ and } H'\}\ .$$

Now recalling the definition 6.4 of the Jung constant and replacing $\subset$ by $\supset$, we define the *Blaschke constant* B_C of C by

$$B_C := \sup\{\sigma > 0\text{: every convex body } Y \text{ of } C\text{-width 2 contains some translate of } \sigma C\}\ .$$

The polar counterpart of 6.5 is

6.19. *If C is a convex body in R^n, then $B_C \geqq 2(n+1)^{-1}$ when C is centrally symmetric, while $B_C \geqq 1/n$ in any case.*

Proof. By 6.18, the widest simplex T which can be circumscribed to C has width $2/B_C$. If $C = -C$, at least one of the hyperplanes bounding T will be no farther from the origin than the centroid of T. Thus $\text{width}_C T \leqq n+1$ and $B_C \geqq 2(n+1)^{-1}$. For the second half of 6.19, we employ an argument polar to one used by Grünbaum [8] for J_C. Since (with $C^* = (C + (-C))/2$) $\|\ \|_{C^*} = \|\ \|_C$, it follows from the inequality already established and from

the definition of J_{C^*} that each convex body of C-width 2 contains a translate of $2(n+1)^{-1}J_{C^*}^{-1}C$. But $J_{C^*} \leqq 2n(n+1)^{-1}$, and consequently $B_C \geqq 1/n$.

The first part of 6.19 was proved initially by Leichtweiss [1] and later also by Eggleston [2]. For the symmetric case[7] Leichtweiss showed that $B_C = 2(n+1)^{-1}$ if and only if there is a simplex T such that $2C \subset T + (-T)$ and $(n+1)T$ is circumscribed to $2C$ (implying that the origin is the centroid of T); this T may serve as Y and vice-versa (up to translation). Hence for general C, $B_C = 1/n$ exactly when C is a simplex (with Y a translate of $-C$).

Recall that J_C attains its lower bound 1 exactly when C is a parallelotope. The polar situation is quite different, for Eggleston [2] proved

6.20. *$B_C = 2/3$ for every centrally symmetric convex body C in E^2.*

When $n = 3$, Leichtweiss's conditions are easily verified for parallelotopes, affinely regular octahedra, elliptic cylinders, and centrally symmetric double cones with convex base.

For higher n, exact upper bounds are not known; so far as we know, they may even be attained for $C = B^n$ (6.21). For this Euclidean case the Blaschke constant is the width of the regular simplex circumscribed to B^n, where the parallel supporting hyperplanes realizing the width each contain about half the vertices. This case was treated by Blaschke [1] for $n = 2$ and in the general case by Steinhagen [1], whose calculations were later simplified by Gericke [1]. The result is as follows:

6.21. *If B^n is the n-dimensional Euclidean cell, then*

$$B_{B^n} = \begin{cases} \sqrt{n+2}/(n+1) & \text{for even } n\,, \\ 1/\sqrt{n} & \text{for odd } n\,. \end{cases}$$

As Steinhagen remarked, for $n \geqq 3$ there are many subsets of a regular simplex $T \subset E^n$ having the same Euclidean width and the same insphere. Nevertheless, it should be possible to conclude from his or Gericke's calculations that every extremal set (for B_{B^n}) lies in a regular simplex of the same width and inradius.

In his paper devoted to B_{B^2}, Blaschke [1] also noted the following consequence of Jung's theorem:

6.22. *Every set of constant width 2 in E^n contains a sphere of radius $2 - \sqrt{2n(n+1)^{-1}}$.*

An apparently difficult problem is to determine the largest number r_n such that every convex body of constant width 2 in E^n contains a *hemisphere* of radius r_n. Clearly $r_n < 1$ for $n \geqq 3$, but it seems probable that $r_2 = 1$. A partial result in this direction is given by Besicovitch [2].

[7] In fact he does not assume symmetry. His "Minkowskische Dicke e" would be "width $_{\mathrm{conv}(C\cup -(-C))}$" in our notation. In similarity to the distinction between J_C^* and J_C in 6.4 and 6.5, his result could be stated as follows: $B_C^* \geqq 2(n+1)^{-1}$ for every convex body C with $0 \in \operatorname{int} C$. Since $C \cup (-C)$ is symmetric and contains C, the general case of Leichtweiss's theorem is immediate from the symmetric one.

Now we shall define the polar correspondents of the expansion constant E_C (6.6) and the constant $\delta(C)$ (6.11). Let C be a convex body in E^n with $C = -C$. For a set $X \subset E^n$, let $X \sim C := \{x : x + C \subset X\}$. Then in analogy with 6.6, we give another description of the Blaschke constant and simultaneously define the *contraction constant*.

6.23. *For a convex body $C \subset R^n$ with $C = -C$, the Blaschke constant B_C [contraction constant S_C] is the largest number σ such that whenever $\mathscr{H}$ is a family of closed halfspaces in R^n with $\pi\mathscr{H} \neq \emptyset$ and each two members of $\{H \sim C : H \in \mathscr{H}\}[\{H \sim \alpha_B C : H \in \mathscr{H}\}]$ have a common point, then the contracted family $\{H \sim \sigma C : H \in \mathscr{H}\}$ $[\{H \sim \sigma\alpha_B C : H \in \mathscr{H}\}]$ has nonempty intersection.*

We ask for lower bounds on S_C and for the value of S_{B^n}.

Finally, we denote by $\bar{\delta}(C)$ the smallest number k such that whenever Y is a convex body of C-width 2, then C can be covered by k translates of Y. We conjecture that $\bar{\delta}(B^2) = 3$, consideration of a regular triangle showing that $\bar{\delta}(B^2) > 2$. (For other related questions, see the report of Grünbaum [17] on Borsuk's problem.) Simple computation comparing the volume of B^n with that of a regular simplex of width 2 in E^n shows that $\bar{\delta}(B^n) > n + 1$ for all large n.

7. **Intersection theorems for special families.** For a set $X \subset R^n$ and a group G of transformations of R^n onto itself, GX will denote the family $\{gX : g \in G\}$. Of special interest are the group T^n of all translations in R^n and the group H^n of all positive homotheties; these will often be denoted simply by T and H. The present section summarizes known results concerning the Hanner- and Gallai-numbers (defined at the end of § 4) of families TC and HC for various convex bodies C.

Hanner [1] proved the following basic theorem:

7.1. *If C is a convex body in R^n, then*

$$\beta_2(TC) = \beta_2(HC) \begin{cases} \text{is infinite when } C \text{ is a parallelotope;} \\ = 3 \text{ when } C \text{ is not a parallelotope but is a centrally} \\ \quad \text{symmetric polyhedron in which every two dis-} \\ \quad \text{joint maximal faces are parallel;} \\ = 2 \text{ in all other cases.} \end{cases}$$

Thus C is a parallelotope if and only if each family of pairwise intersecting translates of C has nonempty intersection. Hanner also proves some facts about the special polyhedra C for which $\beta_2(TC) = 3$ (called *Hanner bodies* in 6.8+) and describes a general construction for many of them. There are no Hanner bodies in E^2 and the only Hanner bodies in E^3 are those which are affinely equivalent to the regular octahedron.

That $\beta_2(T^nC)$ is finite (and hence $\leq n$) for every non-parallelotope C was proved previously by Sz.-Nagy [1]. For $j > 2$, the literature contains no characterization of convex bodies C having $\beta_j(T^nC) \geq \aleph_0$.

The work of Hanner [1] and Sz.-Nagy [1] was inspired by a paper of Nachbin

[1] concerning normed linear spaces E with the following *extension property*: whenever Y is a linear subspace of a normed linear space Z and η is a continuous linear transformation of Y into E, then η can be extended to a continuous linear transformation ζ of Z into E with $\|\zeta\| = \|\eta\|$. One of Nachbin's results was as follows:

7.2. *For a normed linear space E with unit cell U, the following three assertions are equivalent:*

E has the extension property;

every family of pairwise intersecting cells in E has nonempty intersection (i.e., $\beta_2(HU) >$ card HU);

E is equivalent to the space of all continuous real functions over an extremally disconnected compact Hausdorff space.

Part of Nachbin's proof assumed U to have an extreme point, but this assumption was removed by Kelley [1]. Similar results were later obtained by Aronszajn-Panitchpakdi [1] for more general metric spaces. See also Grünbaum [12], Nachbin [2].

The fact that $\alpha(TC) = 2$ (equivalent to $\beta_2(TC) >$ card TC) if C is a parallelotope is generalized by the following, whose proof uses projections in an obvious way.

7.3. *If X_1 and X_2 are nonempty sets in two linear subspaces R^{n_1} and R^{n_2} which span $R^{n_1+n_2}$, then*

$$\alpha(T^{n_1+n_2}(X_1 + X_2)) = \max\{\alpha(T^{n_1}X_1), \alpha(T^{n_2}X_2)\}$$
$$\leqq \alpha(H^{n_1+n_2}(X_1 + X_2)) = \max\{\alpha(H^{n_1}X_1), \alpha(H^{n_2}X_2)\}$$
$$(\leqq 1 + \max\{n_1, n_2\} \text{ when the } X_i \text{ are convex}).$$

Nachbin [2] and others have asked whether $\alpha(H^nC) = n + 1$ for every convex body C in R^n which is not a cartesian sum. A counterexample (previously unpublished) was found by Danzer. Let K be a convex body in R^{n-1} and let f be a real-valued function on $[0, 1]$ which is non-negative, antitone, and upper semicontinuous. Define

$$C := \bigcup_{\alpha \in [0,1]} \{(x, \alpha) : x \in f(\alpha)K\} \subset R^{n-1} + R^1 = R^n ;$$

then $\alpha(H^nC) = \alpha(H^{n-1}K) + 1$. In particular, $\alpha(H^3P) = 3$ when P is a pyramid in E^3 whose base is a square.

Among all convex bodies in R^n, the parallelotopes are characterized by an intersection property of their translates (7.1). The same is true of simplexes, as was proved by Rogers and Shephard [1]:

7.4. *A convex body C in R^n is a simplex if and only if the intersection of each pair of translates of C is empty, onepointed, or a (positive) homothet of C.*

An equivalent condition is that $C_1 \cap C_2 \in HC$ whenever $C_1 \in HC$, $C_2 \in HC$, and $C_1 \cap C_2$ includes more than one point. This was used by Choquet [1; 2] to define infinite-dimensional simplexes. (See Bauer [2] and Kendall [1] for additional details.)

Before discussing the problem of Gallai, we should mention once more the problem of Hadwiger-Debrunner [2], described earlier in 4.4 ff. In addition to the number $J(r, s, n)$ defined there with respect to the family $\mathscr{C}^n$ of all convex sets in R^n, one may consider $J_{\mathscr{F}}(r, s, n)$ defined in the same way for a subfamily $\mathscr{F}$ of $\mathscr{C}^n$. Hadwiger-Debrunner [2] observe that if $\mathscr{P}^n$ is the family of all parallelotopes in R^n with edges parallel to the coordinate axes, then $J_{\mathscr{P}^n}(r, s, n) \leqq \binom{r-s+n}{n}$; further, $J_{\mathscr{P}^n}(r, s, n) = r-s+1$ when $ns \geqq (n-1)r+n$. It would be interesting to study the numbers $J_{\mathscr{F}}(r, s, n)$ for families $\mathscr{F}$ of the form T^nC or H^nC.

* * * * * * * * *

The oldest question of Gallai type is that of T. Gallai, apparently first published in the book of Fejes Tóth [3] (cf. Hadwiger [13]): What is the smallest number of needles required to pierce all members of any family $\mathscr{F} := \{B_\iota : \iota \in I\}$ of pairwise intersecting circular discs in E^2? In our notation, this asks for the value of $\gamma_2(HB^2)$. From Figure 7, P. Ungár and G. Szekeres

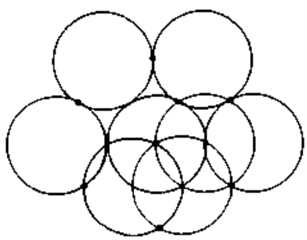


FIGURE 7

concluded that $\gamma_2(HB^2) \leqq 7$. In fact, seven points are enough even when, instead of $\mathscr{F} \in \mathfrak{D}_2$, only $B_\iota \cap B_0 \neq \emptyset$ for all $\iota \in I$ is assumed, where B_0 is a smallest disc in $\mathscr{F}$. Under this weaker assumption, 7 is the best number. Later A. Heppes proved $\gamma_2(HB^2) \leqq 6$. It was reduced to 5 by L. Stachó (L. Szentmártony), and by Danzer to 4, thus settling the question as examples show. All proofs are still unpublished; one example is given by Grünbaum [9], a related remark by Schopp [1].

For guidance in generalizing the original Gallai problem, we review the following facts:

(a) It seems natural to consider only families of convex sets, since otherwise the Gallai-numbers may be infinite even for families of translates (Danzer [3]).

(b) Clearly $\gamma_j(\mathscr{F}) = 1$ when $j \geqq \alpha(\mathscr{F})$, and in particular when $\mathscr{F}$ consists of convex sets in R^n and $j \geqq n+1$.

(c) If $\mathscr{P}^n$ is the family of all parallelotopes with edges parallel to the coordinate axes in R^n, then $\gamma_2(\mathscr{P}^n) = 1$. The condition $\gamma_2(TC) = 1$ characterizes the parallelotopes among all convex bodies C in R^n, and $\gamma_2(HC) = 1$ characterizes the parallelotopes among compact sets C. (Probably $\gamma_2(TC) = 1$

characterizes the parallelotopes among connected compact sets.)

(d) For families of the form GC, where G is a group of linear transformations in R^n with $G \supset T^n$, it seems clear that finite Gallai numbers cannot be expected unless $G \subset H^n$ or C is very special.

In view of the above facts, we restrict our attention to the numbers $\gamma_j(T^nC)$ and $\gamma_j(H^nC)$ where C is a convex body in R^n (but not a parallelotope) and $2 \leqq j \leqq n$. The families of translates will be treated first (7.5–7.8) and then those of homothets. Some related questions on covering and on families of sets adjacent to a given set are discussed at the end of the section.

For the application of 6.3 to Gallai's problem, some special notation is convenient. Suppose $r > 0$, j is a natural number, Y is a bounded set in R^n, and C is a convex body in R^n. Then we define

$$[Y/C]: = \min\{k\colon Y \text{ can be covered by } k \text{ translates of } C\}\,,$$

called the *covering number* of Y with respect to C;

$$\operatorname{diam}_C^j Y: = \inf\left\{2\alpha \geqq 0\colon \begin{array}{l}\text{each } j+1 \text{ points of } Y \text{ can be}\\ \text{covered by a translate of } \alpha C\end{array}\right\},$$

called the jth *C-diameter* of Y; and

$$\delta^j(C, r): = \max\{[Y/C]: Y \subset R^n \text{ with } \operatorname{diam}_C^j Y \leqq 2r\}\,.$$

We shall write $\delta^j(C)$ for $\delta^j(C, 1)$, whence $\delta^1(C) = \delta(C)$ as defined in § 6. From 6.2 it follows that for all $j \geqq n$, $\operatorname{diam}_C^j Y = \operatorname{diam}_C^n Y$, the circumdiameter of Y with respect to C. Although $\operatorname{diam}_C^j Y = \operatorname{diam}_{-C}^j Y$ for all C and Y when $j = 1$, this is not true in general when $j > 1$; but of course $\operatorname{diam}_C^j Y = \operatorname{diam}_{-C}^j(-Y)$ and thus $\delta^j(C, r) = \delta^j(-C, r)$ for all j, C and r.

In conjunction with 6.3, the above definitions lead at once to the following results:

7.5. *If C is a convex body in R^n, $F \subset R^n$, and $\mathscr{F}$ is the family of translates $\{x + C : x \in F\}$, then $\mathscr{F} \in \mathfrak{D}_j$ if and only if* $\operatorname{diam}_{-C}^{j-1} F \leqq 2$.

7.6. *For a convex body C in R^n, the jth Gallai-number of the family TC of all translates of C is given by*

$$\gamma_j(TC) = \delta^{j-1}(-C) = \delta^{j-1}(C)\,,$$

hence is not greater than the number of translates of C needed to cover a translate expanded by the jth Jung constant of C; i.e.,

$$\gamma_j(TC) \leqq [(J_C^j)C/C]\,.$$

(The Jung constants J_C^j were defined in 6.7+.)

Thus the Gallai problem for translates is equivalent to a covering problem, evaluation of the numbers $\delta^{j-1}(C)$. For general j, there are significant results only for the Euclidean case $C = B^n$, and these are derived from Danzer's results 6.8:

7.7. $\delta^{j-1}(B^n) \leqq \left[\left(\frac{nj}{(n+1)(j-1)}\right)^{1/2} B^n/B^n\right] \leqq \left(\frac{(n+2)^3}{3}\left(\frac{j+\sqrt{j}}{j-1}\right)^{n-1}\right)^{1/2}$. [8]

(See 6.12 for the case $j = 2$; for the methods used and for values of j close to $n + 1$, see 7.14 ff.)

For more general C, the main results are due to Grünbaum and are restricted to the case $j = 1$:

7.8. *For a convex body* C,
$\delta^1(C) = 2$ *when* C *is an affinely regular hexagon in* R^2;
$\delta^1(C) \leqq 3$ *when* $C = -C \subset R^2$;
$\delta^1(C) \geqq n + 1$ *when* $C = -C \subset R^n$ *and* C *is strictly convex.*

The second result is in Grünbaum [3], the others in Grünbaum [9] for $n = 2$. Beyond this, almost nothing is known. On the one hand, we know of no n-dimensional C for which $\delta^1(C) > n + 1$. On the other hand, for general C and n we have only very rough upper bounds on the numbers $\gamma_j(T^nC)$ and even for large values of j no explicit numerical bounds on $\gamma_j(T^nC)$ better than those stated below for $\gamma_2(H^nC)$.

Very little is known about the relative magnitudes of $\gamma_j(TC)$ and $\gamma_j(TC^*)$, where C^* is the Minkowski symmetrization of C ($C^* := (C + (-C))/2$). Consider a set F and the corresponding families of translates, $\mathscr{F} = \{x + C : x \in F\}$ and $\mathscr{F}^* = \{x + C^* : x \in F\}$. An elementary calculation shows

(*) *when* $j = 2$, $\mathscr{F} \in \mathfrak{D}_j$ *if and only if* $\mathscr{F}^* \in \mathfrak{D}_j$,

but for any higher j neither of these statements implies the other. Hence no inequality between $\delta^{j-1}(C)$ and $\delta^{j-1}(C^*)$ is trivial, although we do not know of any C for which $\delta^{j-1}(C^*) > \delta^{j-1}(C)$. Note that for a triangle S in R^2, $\delta^1(S) = 3 > 2 = \delta^1(S^*)$.

We turn now to Gallai's problem for families of homothets, where even less is known. The geometrical situation is so complicated that it seems impossible to find an *equivalent* covering problem. However, two different approaches by means of covering do lead to rough upper bounds.

One possibility is to generalize the approach of Ungár and Szekeres mentioned earlier. For a convex body C in R^n, let $\bar{\gamma}(C)$ be the smallest number j which is such that whenever $X \subset R^n$, $\alpha_x \geqq 1$ and $(x + \alpha_x C) \cap C \neq \emptyset$ for each $x \in X$, and $\mathscr{F} := \{C\} \cup \{x + \alpha_x C : x \in X\}$, then $\mathscr{F}$ admits a j-partition (i.e., some j-pointed set intersects all members of $\mathscr{F}$). Then clearly $\gamma_2(HC) \leqq \bar{\gamma}(C)$; and $\gamma_2(H_\pm C) \leqq 2\bar{\gamma}C$ if $H_\pm$ is the group of *all* homotheties in R^n. Further, with $\mathscr{F}$ as described, $z_x \in (\alpha_x + C) \cap C$ and $y_x := \alpha_x^{-1}(x - z_x) + z_x$, it is easily verified that $y_x + C \subset x + \alpha_x C$ and $(y_x + C) \cap C \neq \emptyset$. Thus $\bar{\gamma}(C)$ may be defined alternatively in terms of families of *translates* of C. From this it follows, by reasoning analogous to that in 6.3, that $\bar{\gamma}(C)$ is equal to the covering number $[C + (-C)/C]$. Thus we state

7.9. $\gamma_2(HC) \leqq \bar{\gamma}(C) = [C + (-C)/C]$.

[8] Perhaps this bound is already outdated by the method of Erdös-Rogers [1]. (Cf. the footnote to 7.10.)

The principal known results on the function $\bar{\gamma}$ may be stated as follows (Danzer [**3**; **4**], Grünbaum [**9**]):

7.10. *For a convex body C in R^n, $\bar{\gamma}(C) \leqq 5^n$ when C is centrally symmetric, while $\bar{\gamma}(C) \leqq (4n+1)^n$ in any case.*[9]

7.11. *If C is a convex body in R^2, then $\bar{\gamma}(C) \geqq 7$ when C is strictly convex, while $\bar{\gamma}(C) \leqq 7$ if an affinely regular hexagon A can be inscribed in C in such a way that C admits parallel supporting lines at opposite vertices of A.*

To show that $\bar{\gamma}(C) \leqq 5^n$ when $C = -C$, one obtains a covering of $2C$ by first packing into $5C/2$ as many translates of $C/2$ as possible, and then expanding each of them by a factor 2. For general C, assume the centroid to be at the origin and let $K := C \cap (-C)$. Then $C \subset nK$ by the result of Minkowski [**1**] and Radon [**1**] (or use 2.7) and hence $C + (-C) \subset 2nK$. Covering $2nK$ by translates of K leads to the inequality $\bar{\gamma}(C) \leqq (4n+1)^n$. The first inequality in 7.11 follows from the possibility of arranging six translates of C so that they are all adjacent to C and are cyclically adjacent among themselves. For detailed proofs of 7.10–7.11, as well as for relevant examples and values of $\gamma_2(HC)$ for a few special $C \subset R^2$, see Danzer [**2**; **4**]; for the case $C = -C$, see also Grünbaum [**9**]. Danzer [**4**] gives conditions on a family $\mathscr{F}$ in R^2 which imply the existence of a 7-pointed set intersecting the *interior* of each member of $\mathscr{F}$.

For a Euclidean cell K, the idea of studying $\gamma_2(K)$ by means of $\bar{\gamma}(K)$ was extended by Danzer [**3**] to $\gamma_j(K)$. With the aid of a general theorem on intersections of metric cells (stated in 9.9 below), he proved the following crucial lemma:

7.12. *Suppose $2 \leqq j \leqq n+1$ and $\mathscr{K}$ is a family of Euclidean cells in E^n with $\mathscr{K} \in \mathfrak{D}_j$. Let C_0 be an intersection of $j-1$ members of $\mathscr{K}$ which has smallest (Euclidean) diameter among all such intersections, and suppose*

$$(^*) \qquad K \cap C_0 \neq \emptyset \quad (\text{for all } K \in \mathscr{K}) .$$

Then E^n contains a flat F of dimension $n+2-j$ such that
$F \cap C_0$ is a Euclidean cell;
$\operatorname{diam} C_0 = \operatorname{diam}(F \cap C_0) \leqq \operatorname{diam}(F \cap K)$ *for all* $K \in \mathscr{K}$;
$(F \cap K) \cap C_0 \neq \emptyset$ *for all* $K \in \mathscr{K}$.

In other words, if $\mathscr{K} \in \mathfrak{D}_j$ and $(^*)$ holds, then intersection with a suitable flat will reduce the dimension by $j-2$ and yield again a family of Euclidean cells in which each meets the smallest one. Consequently,

7.13. $\gamma_j(HB^n) \leqq \bar{\gamma}(B^{n+2-j})$.

[9] As pointed out by C.A. Rogers in a letter to the authors, these bounds may be replaced by $\bar{\gamma}(C) \leqq 3^n \delta_n$ in the symmetric case and by $\bar{\gamma}(C) \leqq 3^{n+1}2^n(n+1)^{-1}\delta_n$ in general, where $\delta_n := n \log n + n \log\log n + 5n$. Here δ_n is the density always obtainable by covering R^n by translates of a convex body (Rogers [**1**]), while $3 \cdot 2^n(n+1)^{-1}$ is an estimate for the number $\operatorname{vol}((1/3)C + (2/3)(-C))/\operatorname{vol}(C)$ obtained by consideration of the $(n+1)$-dimensional convex body associated with C (Rogers-Shephard [**2**]).

It would be interesting to know whether, for $1 \leqq i \leqq j-2$ and for every family $\mathscr{K}$ of Euclidean cells in E^n with $\mathscr{K} \in \mathfrak{D}_j$, there exists a flat F of deficiency i in E^n such that the family of intersections $\mathscr{G} := \{F \cap K : K \in \mathscr{K}\}$ has property $\mathfrak{D}_{j-i}$ (or at least $\mathfrak{D}_{j-i-1}$) and some smallest member of $\mathscr{G}$ intersects all members of $\mathscr{G}$. In particular, what about the case $i = j-3$?

A second and simpler approach to the Gallai problem for families HC was recently developed by Danzer (as yet unpublished). Consider a family $\mathscr{F} := \{x + \alpha_x C : x \in X\}$ of homothets of C with $C \in \mathscr{F}$ and with always $\alpha_x \geqq 1$, and suppose $\varepsilon > 0$. For each k, $\mathscr{F}$ can be split into k subfamilies,

$$\mathscr{F}_i(k, \varepsilon) := \{x + \alpha_x C : x \in X, (1+\varepsilon)^{i-1} \leqq \alpha_x < (1+\varepsilon)^i\} \qquad (1 \leqq i \leqq k),$$

and a remainder $\mathscr{F}'(k, \varepsilon)$ consisting of rather large cells. If $\mathscr{F} \in \mathfrak{D}_j$ then each of the k families $\mathscr{F}_i(k, \varepsilon)$ can be pierced by $\delta^{j-1}(C, 1+\varepsilon)$ or fewer points, while $\mathscr{F}'(k, \varepsilon)$ can be pierced by a rather small number μ of points. (If C is smooth and $(1+\varepsilon)^k$ is large enough, $\mu = n+1$.) It follows that

$$\gamma_j(HC) \leqq k\delta^{j-1}(C, 1+\varepsilon) + \mu \leqq k[(1+\varepsilon)(J_0^j)C/C] + \mu .$$

For large n this should yield much better results than the inequalities mentioned above, except when $C = B^n$ and j is close to n. In particular, this method leads to the inequality

$$\gamma_j(HB^n) \leqq k\left[(1+\varepsilon)\left(\frac{nj}{(n+1)(j-1)}\right)^{1/2} B^n \Big/ B^n\right] + [(1 + (1+\varepsilon)^{-k})B^n/B^n] ,$$

where the best numerical results are obtained for $(1+\varepsilon)\sqrt{nj(n+1)^{-1}(j-1)^{-1}} \approx 1 + (1+\varepsilon)^{-k}$.

Of course all the upper bounds for $\gamma_j(HB^n)$ which have been mentioned here are very rough. We know of no example which contradicts $\gamma_j(HB^n) \leqq n + 4 - j$ and also of no C for which $\gamma_j(TC) > n + 3 - j$. As to lower bounds for $\gamma_j(HC)$, nothing is known for $j > 2$.

* * * * * * * * *

In attempting to find values or upper bounds for the Gallai numbers $\gamma_j(T^nC)$ and $\gamma_j(H^nC)$, we were led to various covering problems. For general C and n, difficulty is caused by the fact that the bodies to be covered by translates of C are themselves not much larger than C.

For the special case of B^n, there is more hope of improvement. Recall from 7.6 that $\gamma_j(TC) = \delta^{j-1}(C) \leqq [(J_0^j)C/C]$, whence (7.7)

$$\gamma_j(TB^n) \leqq \left[\left(\frac{nj}{(n+1)(j-1)}\right)^{1/2} B^n \Big/ B^n\right].$$

And by 7.9, $\gamma_2(HC) \leqq \bar{\gamma}(C) = [C + (-C)/C]$, whence

$$\gamma_2(HB^n) \leqq [2B^n/B^n] .$$

Thus we are interested in finding "good" coverings of rB^n by translates of B^n, especially for $1 < r \leqq \sqrt{2}$ and for $r = 2$. As Danzer observed, the task is (almost trivially) equivalent to a similar one on the sphere S^{n-1}:

7.14. *If $\zeta_n(\alpha)$ is the minimal number of spherical caps of (spherical) radius α that will cover the unit sphere S^{n-1} in E^n, then*

$$[(\operatorname{cosec}\alpha)B^n/B^n]\begin{cases} = \zeta_n(\alpha) & \text{for } \pi/2 > \alpha \geqq \pi/4\,, \\ \leqq \zeta_n(\alpha) + 1 & \text{for } \pi/4 > \alpha \geqq \pi/6\,. \end{cases}$$

For small n and specific values of α, one may use particular coverings to get upper bounds. For example, Danzer [3] shows

7.15. $\zeta_2(\pi/6)=6$, $\zeta_3(\pi/6)\leqq 20$, *and* $\zeta_4(\pi/6)\leqq 70$, *whence* $\bar{\gamma}(B^3)\leqq 21$ *and* $\bar{\gamma}(B^4)\leqq 71$.

(See 7.9 – for definition of $\bar{\gamma}$.)

For larger n and arbitrary α one may instead estimate *packings* of radius $\alpha/2$ by use of Blichfeldt's method (see Rankin [1]). This yields (Danzer [3]):

7.16. *For* $n \geqq 3$ *and* $1 < r \leqq 2$,

$$[rB^n/B^n] < \frac{1}{2}((n-2)(r^2-1)^{1/2}+3r)\left(\frac{(2n-1)\pi}{r(r^2-1)^{1/2}}\right)^{1/2}(r(r^2-1)^{1/2}+r^2)^{(n-1)/2}\,.$$ [10]

By a theorem of Rogers [2], the density of a packing of congruent cells in E^n cannot exceed the density in a regular simplex with the vertices being centers of cells. The same result for packings in S^{n-1} would lead to a slight improvement of 7.16. (This result was proved by Fejes Tóth [2; 3] for $n = 3$, but is unknown in general.)

Lower bounds for $[rB^n/B^n]$ are especially interesting for $r = 2$, since we know so little about $\gamma_2(HB^n)$. Dividing the $(n-1)$-measure of S^{n-1} by that of a spherical cap of radius $\pi/6$, Danzer [3] showed that for $n \geqq 3, \zeta_n(\pi/6) > (3n\pi/2)^{1/2}2^{n-1}$. By Coxeter-Few-Rogers [1], the density of a *covering* of E^n by congruent cells cannot be less than the corresponding density in a regular simplex. If the same were known for coverings of S^{n-1}, this inequality could be improved by a factor of about $n/2$. (For $n = 3$ the result was proved by Fejes Tóth [1; 3], though it is uncertain in general. It follows in particular that $\zeta_3(\pi/6) \geqq 19$.)

Hadwiger [11] asked how many translates $C_\iota (\iota \in I)$ of a convex body C in E^n can be arranged so that each intersects C but no two of them have common interior points. (The similar problem with $C_\iota \cap C_\lambda \neq \emptyset$ rather than $C_\iota \cap C \neq \emptyset$ was treated by Danzer-Grünbaum [1].) From $(_*)$ in 7.8+ it follows that the maximum number is the same for C as for C^*, and then a simple argument shows that the maximum is attained only if C itself is among the C_ι. Thus Hadwiger's number is equal to $\bar{\gamma}(C) + 1$, where $\bar{\gamma}(C)$ is the maximum number of translates of C which can be adjacent to C while having no interior points in common with each other. (Two sets are *adjacent* provided their closures meet but not their interiors.) When C is smooth, $\bar{\gamma}(C)$ serves as a trivial lower bound for $\bar{\gamma}(C)$.

The following is known:

[10] Perhaps this bound is already outdated by the method of Erdös-Rogers [1]. (Cf. the footnote to 7.10.)

7.17. *For every convex body C in R^n, $n^2 + n \leqq \tilde{\gamma}(C) = \tilde{\gamma}(C^*) \leqq 3^n - 1$. Further, C is a parallelotope if and only if $\tilde{\gamma}(C) = 3^n - 1$.*

The inequality $n^2 + n \leqq \tilde{\gamma}(C)$ was proved by Grünbaum [15], who observed that it is always possible to have $n + 1$ translates $x_i + C$ *pairwise adjacent* to each other, and then the $n^2 + n$ sets $x_i - x_j + C$ $(i \neq j)$ are in the desired position with respect to C. (See also Swinnerton-Dyer [1].) The inequality $\tilde{\gamma}(C) \leqq 3^n - 1$, given by Hlawka [1], Hadwiger [11], and Groemer [1], follows at once from the observation that

$$1 + \tilde{\gamma}(C) = 1 + \tilde{\gamma}(C^*) \leqq (\mathrm{vol}\, 3C^*)/(\mathrm{vol}\, C^*) .$$

The characterization of parallelotopes is due to Grünbaum [15] for $n = 2$ and to Groemer [1] for general n. (Groemer [2] corrects an erroneous remark of Grünbaum [15].) There seems to be no general characterization of those C for which $\tilde{\gamma}(C) = n^2 + n$, though Grünbaum [15] shows that $\tilde{\gamma}(C) = 6$ for *every* C in R^2 which is not a parallelogram. Grünbaum [15] conjectures that $\tilde{\gamma}(C)$ is always an even number.

It would be interesting to study the numbers $\tilde{\gamma}(C)$ for sets C in R^n assumed merely to be homeomorphic with B^n. This is done for $n = 2$ by Halberg-Levin-Straus [1], who show that even without the assumption of convexity, $\tilde{\gamma}(C) \geqq 6$.

The problem of finding $\tilde{\gamma}(B^n)$ is famous and of long standing. For a discussion of it, see Fejes Tóth [3] for $n = 3$, and the report of Coxeter [1] in this volume for general n. At present the best estimates for $\tilde{\gamma}(B^n)$ are obtained by the methods described above for $\tilde{\gamma}(B^n)$. Substituting $r = (4/3)^{1/2}$ (corresponding to $\alpha/2 = \pi/6$) in the inequality 7.16 to obtain an upper bound, and computing a quotient of volumes for a lower bound, we obtain

$$7.18. \qquad \frac{\sqrt{(2n + 11)\pi}}{2}\left(\frac{4}{3}\right)^{(n-1)/2} \leqq \tilde{\gamma}(B^n) \leqq \frac{n+4}{4}\sqrt{\frac{(2n-1)\pi}{2}}\, 2^{(n-1)/2} .$$

8. **Other intersection theorems.** The material summarized in this section makes little contact with Helly's theorem, but it does treat intersection properties of convex sets in R^n. One group of results owes its initial stimulus to coloring problems (see also 8.6–8.8). Two convex bodies in R^n are called *neighbors* if their intersection is of dimension $n - 1$, and a family of convex bodies is *neighborly* provided each two of its members are neighbors. For an obvious reason, the neighborly families require special attention in connection with coloring problems. A neighborly family of convex (or much more general) bodies in E^2 can have at most four members. On the other hand, Tietze [1] answered a question of Stäckel by constructing in E^3 an infinite neighborly family of convex polyhedra. (See Tietze [2] for a survey of this and related questions.) M. Crum posed the problem independently about forty years after its solution by Tietze, and a second construction was given by Besicovitch [1]. A more detailed study of intersection properties of convex polyhedra in R^n was made by Rado [3] and Eggleston [1], who proved the

following (the first statement in each case being that of Rado, the second that of Eggleston).

8.1. *When the integer m is $\leqq (n+1)/2$, R^n contains an infinite family $\mathscr{P}$ of convex polyhedra such that for all j with $1 \leqq j \leqq m$, each j members of $\mathscr{P}$ have an $(n-j+1)$-dimensional intersection. When $m > (n+1)/2$, such a family does not exist in R^n.*

8.2. *R^n contains a family of $n+2$ convex polyhedra, each j having $(n-j+1)$-dimensional intersection for all j with $1 \leqq j \leqq n$. Such a family in R^n cannot have $n+3$ members.*

These results on neighborly families are closely connected with the neighborly polyhedra of Gale [3; 5] (see also Carathéodory [2], Motzkin [3], and 3.9 above). For example, let P_k be a convex polyhedron in R^4 which has k vertices such that each segment joining two vertices is an edge of P_k. Then the dual polyhedron Q_k has k 3-faces, each two of which intersect in a 2-face of Q_k. Let K be a 3-face of Q_k, z a point exterior to K but very close to a (relatively) interior point of K, and from z as center, project all the other faces of Q_k into K. The resulting configuration consists of $k-1$ 3-dimensional polyhedra whose union is the convex polyhedron K, and each two of these polyhedra intersect in a common 2-face.

In the constructions of Tietze, Besicovitch, and Rado there was no restriction on the numbers of vertices or faces of the polyhedra in question. However, Bagemihl [1] asked for the maximum number k of *tetrahedra* in a neighborly family in R^3; he proved that $8 \leqq k \leqq 17$ and conjectured that $k = 8$. Baston [1] has recently proved that $k \leqq 9$, but it remains undecided whether 8 or 9 is the actual value. More generally, the following problem is nearly untouched: For $j \geqq n+1$, determine the maximum number $N(n, j)$ of convex polyhedra having j vertices each which can appear in a neighborly family in R^n. An extension of the reasoning of Bagemihl [1] shows that always $N(n, j) < \infty$. One might ask the same question with the restriction that the polyhedra in question be affinely or combinatorially equivalent to a given one, but even here the literature is silent except for the results of Bagemihl and Baston. (For a related result involving translative equivalence of tetrahedra, see Świerczkowski [1].)

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Another group of results owes its basic notion to the theory of probability. A family $\mathscr{G}$ of subsets of a set E is said to be *independent in E* if for every subfamily $\mathscr{F} \subset \mathscr{G}$

$$\pi\mathscr{F} \cap \pi\{E \sim G : G \in \mathscr{G} \sim \mathscr{F}\} \neq \emptyset .$$

Let us define the *rank* of a family $\mathscr{F}$ of sets as the least upper bound of the cardinalities of independent subfamilies of $\mathscr{F}$. Then the results of Rényi-Rényi-Surányi [1] may be stated as follows.

8.3. *The family of all open parallelotopes in R^n, with edges parallel to the coördinate axes, is of rank $2n$.*

8.4. *The family of all $(n-1)$-dimensional Euclidean spheres in E^n is of rank $n+1$.*

8.5. *If r_j is the rank of the family of all open convex polygonal domains of at most j sides in R^2, then* $\lim_{j\to\infty} r_j/\log j = 1/\log 2$.

The proof of 8.3 is straightforward, while that of 8.4 depends on an estimate of the maximum numbers of parts B_k^n into which E^n can be divided by k spheres. The exact value of B_k^n is tabulated for $1 \leqq n \leqq 10, 1 \leqq k \leqq 10$, and several problems are suggested.

* * * * * * * * *

For the problems of Gallai (§ 7) and of Hadwiger-Debrunner [2] (4.4 above), one seeks conditions on a family $\mathscr{F}$ of convex sets in R^n which assure that $\mathscr{F}$ can be partitioned into m subfamilies, each with nonempty intersection. On the other hand, Bielecki [1], Rado [4], and Asplund-Grünbaum [1] have sought conditions under which $\mathscr{F}$ is *m-colorable*, this meaning that $\mathscr{F}$ can be partitioned into m families of pairwise disjoint sets. For a family $\mathscr{F}$ of sets and for natural numbers p and q, let $\gamma_{p,q}(\mathscr{F})$ denote the smallest natural number r which has the following property: if $\mathscr{G} \subset \mathscr{F}$ and each p-membered subfamily of $\mathscr{G}$ is q-colorable, then $\mathscr{G}$ is r-colorable ($\gamma_{p,q}(\mathscr{F}) := \infty$ when no such r exists). The result of Rado [4] and Bielecki [1] is

8.6. *If $\mathscr{F}$ is the family of all intervals in an ordered set, $\gamma_{k+1,k}(\mathscr{F}) = k$.*

Asplund-Grünbaum [1] prove

8.7. *If $\mathscr{R}$ is the family of all plane rectangles with edges parallel to the coördinate axes, $\gamma_{3,2}(\mathscr{R}) = 6$.*

Further, a graph-coloring theorem of Grötzsch [1] implies

8.8. *For each convex body K in R^2, $\gamma_{3,2}(HK) = 3$ (where HK is the family of all (positive) homothets of K).*

The paper of Asplund and Grünbaum contains other results on colorability of various families, and raises some unsolved problems. The following general problem seems worthy of mention: For what families $\mathscr{F}$ of convex sets in R^n and for what p and q is the coloring number $\gamma_{p,q}(\mathscr{F}) < \infty$? How does it depend on p, q and n? In particular, is $\gamma_{3,2}(\mathscr{F}) < \infty$ when $\mathscr{F}$ is the family of all convex bodies in R^2, or when $\mathscr{F}$ is the family of all parallelepipeds in R^3 with sides parallel to the coördinate axes? (Some partial answers have recently been obtained by Danzer and Grünbaum.)

These "coloring problems" suggest the following extension of the Gallai-type problem formulated near the end of § 4:

Suppose $X, \mathfrak{P}, \kappa, \lambda,$, and conditions on $\mathscr{F}$ are given. Determine the smallest cardinal ψ such that for every family $\mathscr{F}$ which satisfies the given conditions, and for which $\mathscr{G} \subset \mathscr{F}$ and card $\mathscr{G} < \kappa + 1$ imply that $\mathscr{G}$ can be split into μ families each with property $\mathfrak{P}$, it is possible to split $\mathscr{F}$ into ψ subfamilies each having property $\mathfrak{P}_\lambda$.

In the above terms, the result 8.7 asserts that if $\mathfrak{P}$ is the property "the members of the family are pairwise disjoint," $X = R^2$, $\mathscr{F} = \mathscr{R}$, $\kappa = 3$, $\mu = 2$, and $\lambda = \aleph_0$, then $\psi = 6$; and 8.6 asserts that if X is an ordered set, $\mathscr{F} = \mathscr{I}$, $\kappa = k + 1$, $\mu = k$, and $\lambda =$ successor of card X, then $\psi = k$.

An unsolved problem related to Gallai's original geometric problem arises when $X = R^2$, $\mathfrak{P} = \mathfrak{D}$ (nonempty intersection), $\mathscr{F} = TK$ for a convex body $K \subset R^2$, $K = 2k$, $\mu = k$, and $\lambda = \aleph_0$. When K is strictly convex and centrally symmetric, a result of Grünbaum [9] implies that $\psi \geqq 3n$. It is easy to verify that $\psi \leqq 8$ when K is a circle and $k = 2$; Grünbaum conjectures that $\psi = 6$ in this case.

* * * * * * * * *

If φ is a transformation of a metric space (X, ρ) into a metric space (Y, σ), then φ is called *Lipschitzian with constant* λ provided $\sigma(\varphi x, \varphi x') \leqq \lambda\rho(x, x')$ for all $x, x' \in X$; and φ is a *contraction* provided always $\sigma(\varphi x, \varphi x') \leqq \rho(x, x')$. Several authors have studied the problem of extending (to all of X, with preservation of constant) a Lipschitzian transformation which is initially defined only on a subset of X. Two early papers on the subject were those of McShane [1] and Kirszbraun [1]. The method of McShane (and, later, of Banach [1] and Czipszer-Gehér [1]) provided explicit formulae for the extension in certain cases, while the "point-by-point" method of Kirszbraun, Valentine [1; 2; 3], and Mickle [1] used an interesting intersection property. We shall discuss here only contractions, for the general Lipschitzian extension problem can be reduced to that for contractions when Y is a normed linear space.

When $\mathscr{F}$ and $\mathscr{G}$ are families of metric cells in (X, ρ) and (Y, σ) respectively, we shall write $\mathscr{F} \succ \mathscr{G}$ if the families can be indexed simultaneously in such a way that corresponding cells have equal radii and the distance between any two centers from $\mathscr{F}$ is at least that between the corresponding centers from $\mathscr{G}$. Thus with

$$\mathscr{F} := \{V_\rho(x_\iota, r_\iota) : \iota \in I\}, \quad \mathscr{G} := \{V_\sigma(y_\iota, r_\iota) : \iota \in I\},$$
$$\mathscr{F} \succ \mathscr{G} \Longleftrightarrow \rho(x_\iota, x_{\iota'}) \geqq \sigma(y_\iota, y_{\iota'}) \quad \text{for all} \quad \iota, \iota' \in I.$$

Under these circumstances, it is natural to ask whether $\pi\mathscr{F} \neq \emptyset$ implies $\pi\mathscr{G} \neq \emptyset$. That is, if the cells in $\mathscr{F}$ have a common point, should they not again have a common point if we displace their centers so as not to be farther apart than they were before? In fact, the situation is as follows (Valentine [3]):

8.9. *For metric spaces X and Y, the following two statements are equivalent:*
whenever $\mathscr{F}$ and $\mathscr{G}$ are families of metric cells in X and Y respectively, with $\mathscr{F} \succ \mathscr{G}$, then $\pi\mathscr{F} \neq \emptyset$ implies $\pi\mathscr{G} \neq \emptyset$;
whenever $W \subset X$, each contraction of W into Y can be extended to a contraction of X into Y.

From 8.9 it is clear that results of all the authors mentioned above may be interpreted as intersection theorems, though they were not always stated as such. There seems to be no complete picture even of what pairs of Minkowski

spaces X and Y are related as in 8.9, though Zorn observed that the conditions may fail for large classes of such pairs (see Valentine [3]). In any case, the following is known:

8.10. *For each of the following cases, the conditions of* 8.9 *are satisfied:*
a° *X an arbitrary metric space, $Y = E^1$;*
b° *$X = Y = E^n$;*
c° *$X = Y = S^n$ (spherical space).*

8.11. *When X and Y are the same two-dimensional Minkowski space with unit sphere S, the conditions of* 8.9 *are satisfied if and only if S is an ellipse or a parallelogram.*

The result 8.11 is due to Grünbaum [7]. The proof of 8.10a is given by McShane [1], Valentine [1], Banach [1], Mickle [1], and Czipszer-Gehér [1], with various analytical generalizations (e.g., to functions satisfiying a Lipschitz-Hölder condition). From 8.10a it follows that the conditions of 8.9 are also satisfied for arbitrary X when Y is the space of all continuous real functions on an extremally disconnected compact Hausdorff space, and thus in particular when Y is a Minkowski space whose unit cell is a parallelotope (cf. 7.1 and 7.2). Case b° is discussed by Kirszbraun [1], Valentine [3], Mickle [1], Schoenberg [1], and Grünbaum [19], and (as noted by Valentine) extends easily to the case $X = Y =$ complete inner-product space. The case c° is treated by Valentine [3], who also [2] establishes the result for $X = Y = n$-dimensional hyperbolic space. Helly's theorem is employed by Kirszbraun [1] and Valentine [1; 2; 3]; in particular, it is clear that for 8.7b it suffices to consider families consisting of $n + 1$ cells. All the proofs employ analytical notions (inner-products, etc.) even for $X = Y = E^2$, but it would seem worthwhile to seek a more geometrical proof. The simplest proofs for 8.10b are those of Schoenberg [1] and of Grünbaum [19], who proves a generalization (8.12 below) which includes (by suitable choice of x_i, y_i, α and β) both Kirszbraun's theorem and a recent theorem of Minty [1].

8.12. *Suppose $\{x_1, \cdots, x_m\} \subset E^n$, $\{y_1, \cdots, y_m\} \subset E^n$, the inner product $(x_i - x_j, y_i - y_j) \geqq 0$ for all i, j, and α and β are real numbers not both zero. Then there exists $z \in E^n$ such that $(x_i + \alpha z, y_i + \beta z) \geqq 0$ for all i.*

There are some other interesting results and problems which involve contractions in E^n. By considering the case of cells all of the same radius, we conclude from Kirszbraun's theorem that if a set $X \subset E^n$ admits a contraction onto a set Y, then the *circumradius* of Y is not greater than that of X. While there exist X and Y in E^n such that X can be contracted onto Y even though Y has greater *width*, this cannot happen when X and Y are both convex bodies and the contraction is biunique (Gale [4]). Nothing seems to be known about the behavior of *inradii* in this case.

There is an unsolved problem of Helly type concerning continuous contractions in E^n. For ordered k-tuples $(x_1, \cdots, x_k)$ and $(y_1, \cdots, y_k)$ of points in a metric space (M, ρ), a *continuous contraction* of the first onto the second is an ordered k-tuple $(\varphi_1, \cdots, \varphi_k)$ of continuous mappings of $[0, 1]$ into M such that for all

i and j, $\varphi_i(1) = x_i$, $\varphi_i(0) = y_i$, and $\rho(\varphi_i(\alpha), \varphi_j(\alpha)) \leqq \rho(\varphi_i(\beta), \varphi_j(\beta))$ whenever $0 \leqq \alpha \leqq \beta \leqq 1$. The problem is to evaluate the number m_n, defined as the smallest j such that whenever $(x_1, \cdots, x_k)$ and $(y_1, \cdots, y_k)$ are k-tuples from E^n for which each j-tuple of the x_i's admits a continuous contraction onto the corresponding j-tuple of the y_i's, then $(x_1, \cdots, x_k)$ admits a continuous contraction onto $(y_1, \cdots, y_k)$. (If no such j exists, $m_n := \infty$.) It is easy to see that $m_1 = 2$. A. H. Cayford has remarked that from consideration of the positive and negative unit vectors along the n axes in E^n, together with another point far removed from this set, it follows that $m_n \geqq 2n + 1$ when $n \geqq 2$.

A striking problem is that of Thue Poulsen [1] and M. Kneser [1]: When $\mathscr{F}$ and $\mathscr{G}$ are families of unit cells in E^n, does $\mathscr{F} \succ \mathscr{G}$ imply $\text{vol}\, u\mathscr{F} \geqq \text{vol}\, u\mathscr{G}$ (where u indicates union) and $\text{vol}\, \pi\mathscr{F} \leqq \text{vol}\, \pi\mathscr{G}$? An affirmative solution would be useful in connection with Minkowski's theory of surface area. Kirszbraun's theorem gives a bit of information on the problem, and Kneser shows that $\text{vol}\, u\mathscr{G} \leqq 3^n \text{vol}\, u\mathscr{F}$. (For other details see M. Kneser [1] and Hadwiger [9].) It seems clear that in non-Euclidean Minkowski spaces, the answer to the above question is negative. In the two-dimensional case, this follows from 8.11.

For a Minkowski space X with unit cell C, it would be of interest to study the number E_C' defined as the smallest $\sigma > 0$ such that whenever $\mathscr{F}$ and $\mathscr{G}$ are families of homothets of C with $\mathscr{F} \succ \mathscr{G}$ and $\pi\mathscr{F} \neq \emptyset$, then a common point is obtained upon expanding all the members of $\mathscr{G}$ about their centers by the factor σ. The number J_C' is defined similarly for families of translates of C. Clearly $1 \leqq J_C' \leqq J_C$ and $1 \leqq E_C' \leqq E_C$, where J_C and E_C are the Jung constant and the expansion constant of § 6. Is it possible to have $1 < E_C' = E_C$? What is E_C' when C is a Leichtweiss body (cf. 6.5+)?

9. Generalized convexity. The applicability and the intuitive appeal of convexity have led to a wide range of notions of "generalized convexity." For several of them, theorems related to Helly's were either a motive or a by-product of the investigation. For others, it seems probable that no attempt has been made to find analogues of Helly's theorem. In order to encourage such attempts and to facilitate comparison of the various notions which have been considered, our discussion will include several generalizations of convexity which seem at present unrelated to Helly's theorem.

The usual procedure in defining a generalized convexity is to select a property of convex sets in R^n or E^n which is either characteristic of convexity or essential in the proof of some important theorem about convex sets, and to formulate that property or a suitable variant in other settings. Many properties of convex sets are useful for this purpose. We shall first describe in general terms the most important procedures which have been adopted, and then review briefly some of the results obtained in specific cases.

The usual definition of convexity in R^n can be generalized according to the following scheme. In a set X, a family $\mathscr{F}$ of sets is given together with a function η which assigns to each $F \in \mathscr{F}$ a family $\eta(F)$ of subsets of X. A set $K \subset X$ is called *η-convex* provided K contains at least one member of $\eta(F)$

whenever $F \subset K$ and $F \in \mathscr{F}$. This is the most common form of generalization, and appears in most of the cases discussed below. Usually $\mathscr{F}$ is the family of all two-pointed subsets of X and Prenowitz [1], for example, studies convexity in abstract geometries based on the notion of "join." For other possibilities see 9.7 below and Valentine [4], where one such variant is discussed and references to others are given.

Another approach derives from the fact that every convex set in R^n is an intersection of semispaces (Motzkin [1], Hammer [1; 2], Klee [4]), and every closed [resp. open] convex set is an intersection of closed [resp. open] half-spaces. A family $\mathscr{H}$ of subsets of a set X is given, and a set $K \subset X$ is called *$\mathscr{H}$-convex* provided K is the intersection of a subfamily of $\mathscr{H}$. (Instances of this appear in 9.1, 9.3, and 9.9 below. See also Ghika [3].) The definitions of $\mathscr{H}$-convexity can also be expressed in terms of "separation": a set $K \subset X$ is $\mathscr{H}$-convex if and only if for every $x \in X \sim K$ there exists $H \in \mathscr{H}$ such that $K \subset H$ and $x \notin H$. (For a generalization in this direction, see Ellis [1].) The family $\mathscr{H}$ may be defined in terms of a family of functions, or more directly we may be given a family Φ of real-valued functions on X and say that a set $K \subset X$ is *Φ-convex* provided for each $x \in X \sim K$ there exists $\varphi \in \Phi$ such that $\varphi(x) < \inf \varphi K$. This procedure is followed by Fan [1] in his generalization of the Krein-Milman theorem, and enters also in various notions of "regular convexity" (see Berge [1] and his references).

Another approach is based on the fact that a subset of R^n is convex if and only if its intersection with each straight line is connected. Working in R^2 and using in place of the lines a two-parameter family of curves, Drandell [1] obtained interesting results under very weak assumptions. His paper also gives references for previously known results in this direction. (See Skornyakov [1] for a relevant result on curve families in R^2, and Valentine [3], Kuhn [1], Fáry [1], Kosiński [1], and their references for some characterizations of convexity in terms of k-dimensional sections.)

Of course one may generalize, in various ways, the underlying algebraic structure in terms of which convexity is formulated. Many of the combinatorial results of this paper are valid in a linear space over an arbitrary ordered field. Monna [1] discusses convexity in spaces over nonarchimedean ordered fields. Other algebraic formulations are those of Rado [2] and Ghika [1; 2], the former being described briefly in 9.4 below.

One may ignore the surrounding linear space and seek a more intrinsic notion of *barycentric calculus* or *convex space*. This was done by Stone [1; 2], H. Kneser [1], and Nef [1]. Their theories turn out to be embeddable in linear spaces, but might be of independent interest if reformulated to introduce topological as well as algebraic structure.

For an open subset D of a metric space (M, ρ), let δ_D and ν_D denote the inner and outer distance functions of D. These functions are defined on D and M respectively, with $\delta_D(x) := \rho(\{x\}, M \sim D)$ and $\nu_D(x) := \rho(\{x\}, D)$. When $M = E^n$ and ρ is the Euclidean distance, convexity of D is equivalent to that of the function $-\log \delta_D$ and also to that of the function ν_D. This suggests that for a family Φ of functions on subsets of a metric space (M, ρ), one could

say that an open set $K \subset M$ is Φ-convex provided $-\log \delta_K \in \Phi$, or perhaps provided $\nu_K \in \Phi$. The notion of convexity in spaces of several complex variables may be approached in this way (see 9.11). However, a more useful idea suggested by these considerations is that one should try to generalize the sort of "convex structure" which is formed by the convex subsets of R^n together with the convex or concave functions defined over them. This leads in one direction to the theory of complex convexity (9.11) and in another to the abstract minimum principle of Bauer [1], generalizing the Krein-Milman theorem.

Finally, we mention the fact that each new characterization of convexity in R^n may lead to new generalized convexities, and that, conversely, the search for a suitable notion of convexity in a given setting may lead to discovery of new and useful properties of sets which are convex in the classical sense. For an interesting example, see Rådström's work ([1; 2] and 9.10 below) in topological groups.

We proceed now to discuss some specific "convexities," commencing with those which appear to be most closely related to Helly's theorem.

9.1. Spherical convexity. Convexity on the n-dimensional sphere S^n has been considered from various points of view, and is certainly the most significant of the generalized convexities so far as Helly's theorem is concerned. Several different definitions of spherical convexity have been studied, though not always with due regard to the limitations which they impose. For the sake of simplicity, we shall discuss only closed sets, though in most cases this restriction can be avoided.

(S) A set $K \subset S^n$ is *strongly convex* iff it does not contain antipodal points and it contains, with each pair of its points, the small arc of the great circle determined by them. Equivalently, K (being closed) is strongly convex if it does not contain antipodal points and is an intersection of closed hemispheres. Every strongly convex K is an intersection of open hemispheres, and vice versa.

(W) A set $K \subset S^n$ is *weakly convex* iff it contains, with each pair of its points, the small arc or a semicircular arc of a great circle determined by them. Equivalently, K is connected and is an intersection of closed hemispheres.

(R) A set $K \subset S^n$ is *Robinson-convex* iff it contains, with each of its non-antipodal points, the small arc of the great circle determined by them. Equivalently, K is an intersection of closed hemispheres.

(H) A set $K \subset S^n$ is *Horn-convex* iff it contains, with each pair of its non-antipodal points, at least one of the great circle arcs determined by them.

Obviously, every strongly convex set is weakly convex; weak convexity implies Robinson-convexity, and that implies Horn-convexity. The strongly convex sets and the Robinson-convex sets form intersectional families. All strongly convex sets are contractible, as are all the Robinson- or weakly-convex sets except for the great m-spheres $S^m \subset S^n$. According to purpose, one or another definition of spherical convexity may be more appropriate. Horn-

convexity was introduced by Horn [1] in proving his generalization of Helly's theorem. Robinson-convexity was used by Robinson [2] in studying congruence-indices of spherical caps (see also Blumenthal [2]). Weakly convex sets were considered by Santaló [4] among others, though the definition at the beginning of his paper is different and includes only a subclass of the strongly convex sets.

An exhaustive bibliography of papers dealing with spherical convexity would be very extensive. Almost every notion and result on convexity in R^n can be extended to S^n, and in many cases the extension has been at least partly accomplished in connection with needs arising from other problems. As an example, we mention a little-known paper of Vigodsky [1] which proves an analogue of Carathéodory's theorem for S^2. (Vigodsky's main theorem was proved earlier by Fenchel [2].) It seems that Carathéodory's theorem and its variants have not been generalized in full to the spherical case, though such an extension is surely possible.

For certain types of problems, the treatment in S^n is more satisfactory than that in R^n, due to the possibility of dualization in S^n. For example, the analogues in S^n of Jung's theorem [1] on circumspheres and Steinhagen's [1] on inspheres are dual aspects of the same result (Santaló [5]). On the other hand, some results in S^n have no Euclidean analogue. An example of this kind is furnished by the difference between the following two relatives of Jung's theorem, in which connectedness plays an essential role though it is irrelevant in R^n: If a compact subset of S^n has diameter less than $\arccos (n+1)^{-1}$, it lies in a closed hemisphere; a compact subset whose diameter is equal to $\arccos (n+1)^{-1}$ need not lie in any hemisphere (Molnár [4]). If a compact *connected* subset of S^n has diameter $\leq \arccos n^{-1}$, it lies in a closed hemisphere (Grünbaum [14]).

Due to the close and obvious relationship between spherical convexity in S^n and convex cones in R^{n+1}, many results can be interpreted in both the spherical and the Euclidean setting. This fact has been used repeatedly (e.g., Motzkin [1], Robinson [1], Horn [1]).

For spherical analogues of Helly's theorem, the simplest approach seems to be through Helly's topological theorem. From this it follows that the family of all homology cells in S^n has Helly-order $n+2$. Alternatively, the result may be stated as follows: A family $\mathscr{F}$ of homology cells in S^n has nonempty intersection provided each union of $n+2$ members of $\mathscr{F}$ is different from S^n and each intersection of $n+1$ or fewer members of $\mathscr{F}$ is a homology cell. This applies, in particular, to the case of strongly convex sets considered by Molnár [4]. Similar corollaries may be derived for the other types of spherical convexity, but their formulation is somewhat more complicated due to the absence of intersectionality, or contractibility, or both (see Horn [1], Karlin-Shapley [1], Grünbaum [14]).

Other results of Helly-type for S^n may easily be derived from results on convex sets in R^n. For example, the case $j=1$ of Theorem 4.1 (essentially contained in Steinitz's result 3.2) may be reformulated as the following fact, first stated by Robinson [2]: *If $\mathscr{F}$ is a family of at least $2n+2$ Robinson-*

convex sets in S^n *and each* $2n+2$ *members of* $\mathscr{F}$ *have a common point, then* $\pi\mathscr{F} \neq \emptyset$. This and related results were employed by Blumenthal [2] in connection with linear inequalities. For some intersection and covering theorems involving hemispheres, see Hadwiger [4] and Blumenthal [2; 4]; a question raised by Hadwiger is answered by Grünbaum [9].

Horn-convex sets have been studied by Horn [1] and Vincensini [3; 4; 5]. Horn shows that if $1 \leqq k \leqq n+1$ and $\mathscr{F}$ is a family of at least k Horn-convex sets in S^n such that each k members of $\mathscr{F}$ have a common point, then every great $(n-k)$-sphere in S^n lies in a great $(n-k+1)$-sphere which intersects each member of $\mathscr{F}$ (cf. 4.3). For $k = n+1$, this asserts the existence of a point z such that every member of F includes z or its antipode.

9.2. Projective convexity. A set K in the n-dimensional (real) projective space P^n is called *convex* provided it contains, with each pair of its points, exactly one of the two segments determined by these points. Such sets have been considered repeatedly since their introduction by Steinitz [1] and Veblen-Young [1], though most authors have discussed only the equivalence of various definitions. (See de Groot-de Vries [1] and other papers listed by them.) The projectively convex sets are contractible but do not form an intersectional family. The intersection of $k+1$ convex sets in P^n may have up to $\sum_{i=1}^{n}\binom{k}{i}$ components, each of which is projectively convex (Motzkin [1]). On the other hand, if each two of the sets have convex intersection, then so has the entire family. Thus the following is an immediate consequence of Helly's topological theorem: *If a family* $\mathscr{F}$ *of at least* $n+1$ *closed convex sets in* P^n *is such that each two members of* $\mathscr{F}$ *have convex intersection and each* $n+1$ *members have nonempty intersection, then* $\pi\mathscr{F} \neq \emptyset$ (Grünbaum [11]).

For $k \geqq 1$, let $\mathscr{V}(n,k)$ denote the class of all subsets of P^n which are the union of k or fewer pairwise disjoint closed convex sets. It seems probable that $\mathscr{V}(n,k)$ is of finite Helly-order, but apparently this has not been established and surely the exact value even of $\alpha^0(\mathscr{V}(n,2))$ is unknown (cf. 4.11).

For other results related to projective convexity, including some on common transversals, see Fenchel [3], Kuiper [1], de Groot-de Vries [1], Schweppe [1], Gaddum [1], Marchaud [1], Grünbaum [11] and Hare-Gaddum [1].

9.3. Levi's convexity. Like that of Rado discussed in 9.4, this notion was motivated primarily by Helly's theorem. Levi [1] considers a family $\mathscr{C}$ of subsets of a set X, and assumes:

(I) $\pi\mathscr{G} \in \mathscr{C}$ *for all* $\mathscr{G} \subset \mathscr{C}$.

For each set Y lying in some member of $\mathscr{C}$, the $\mathscr{C}$-hull is defined as the intersection of all members of $\mathscr{C}$ which contain Y. The second axiom is

(II_n) *Every* $(n+2)$*-pointed subset of a member of* $\mathscr{C}$ *contains two disjoint sets whose* $\mathscr{C}$*-hulls have a common point.*

Generalizing Radon's proof [2], Levi deduces from the above axioms that if $\mathscr{F}$ is a finite family of at least $n+1$ members of $\mathscr{C}$, and each $n+1$ members of $\mathscr{F}$ have a common point, then $\pi\mathscr{F} \neq \emptyset$. As corollaries he obtains Helly's theorem on convex sets in R^n, its extension to n-dimensional

geometries satisfying Hilbert's axioms of incidence and order (Hilbert [1]), and an intersection theorem in free abelian groups with n generators. (In the last case, $\mathscr{C}$ is the family of all subgroups, the neutral element being omitted from each.)

It would be of interest to study, in a system assumed to satisfy (I) and perhaps other simple "axioms of convexity," the inter-relationships of Radon's property as expressed by (II_n), Helly's property as stated above, and Carathéodory's property expressed as follows: (III_n) *Whenever a point p lies in the $\mathscr{C}$-hull of a set Y, then p is in the $\mathscr{C}$-hull of some at-most-$(n+1)$-pointed subset of Y.*

9.4. Rado's convexity. Rado [2] considers an abelian group A under the action of a commutative ring $\mathscr{R}$ of operators, and sets forth conditions on A and $\mathscr{R}$ which assure the validity of Helly's theorem (with a proper interpretation of "convex set"). His reasoning yields not only Helly's theorem on convex sets in R^n, but also a generalization of a theorem of Stieltjes [1] on arithmetic progressions. With respect to a given coördinatization of R^n, a *lattice* is the set of all points of the form $x_0 + \sum_{i=1}^{n} \alpha_i x_i$, where the x_i's are $n+1$ given points with integral coordinates and the α_i's are arbitrary integers. Rado proves that if $\mathscr{F}$ is a finite family of at least $n+1$ lattices in R^n such that each $n+1$ of them have a common point, then $\pi\mathscr{F} \neq \emptyset$.

9.5. Hyperconvexity. If K is a compact convex set in R^n, a set $A \subset R^n$ is *K-convex* (or *hyperconvex with respect to K*) provided A contains, with each pair of its points, the intersection of all translates of K which contain those points. K-convex sets form an intersectional family and the connected K-convex sets are intersections of translates of K. K-convex sets need not be connected, but attention is usually restricted to their connected components. The study of hyperconvexity was initiated by Mayer [1], and a complete set of references can be found through Pasqualini [1], Blanc [1], and Santaló [6]. Most of the papers treat only the planar case, and under restrictions on K. They deal with support properties, equivalent definitions, hyperconvex hulls, extremal problems, etc.

9.6. Quasiconvexity. Let Λ be a subset of $[0, 1]$. A set $K \subset R^n$ is *Λ-convex* (or *quasiconvex with respect to Λ*) provided $\lambda x + (1-\lambda)y \in K$ whenever $\lambda \in \Lambda$ and $x, y \in K$. This notion was studied by Green and Gustin [1], who give references for the previously considered special cases (such as $\Lambda = \{1/2\}$). A detailed study of even more general concepts is the paper of Motzkin [4].

9.7. Three-point convexity. A set K in R^n is *3-point convex* if it contains, with each three of its points, at least one of the three segments determined by these points. Valentine [4; 5] has made an interesting study of this property, and a more general notion has been formulated by Allen [1]. Hare and Gaddum [1] discovered a connection between 3-point convexity and projective convexity. It seems likely that the family of all 3-point convex subsets of R^n is of finite Helly-order, but this has not been determined. (See also Valentine [6; 7].)

9.8. Order-convexity. A set K in a partially ordered space is *order-convex* provided $y \in K$ whenever $x, z \in K$ with $x < y < z$. For results on extreme points and convex hulls with respect to order-convexity, see Franklin [1]. Apparently the literature contains no results of Helly-type about order-convexity in general partially ordered linear spaces, though in any *linearly* ordered space the family of all order-convex sets is intersectional and has Helly-number 2.

9.9. Metric convexity. The usual definition of metric convexity is due to Menger [1]: A set K in a metric space (M, ρ) is *metrically convex* if for each pair of distinct points $x, z \in K$ there exists a point $y \in K$, different from x and z, such that $\rho(x, z) = \rho(x, y) + \rho(y, z)$. When K is not only convex but also metrically complete, then the points x and z of K must lie in a subset of K which is isometric with a linear interval of length $\rho(x, z)$. A closed subset of E^n (or of any strictly convex Banach space) is metrically convex if and only if it is convex in the usual sense. For geodesic distance on the sphere S^n, a closed set is metrically convex if and only if it is weakly convex in the sense of 9.1.

For various geometric developments involving Menger's and other closely related notions of metric convexity, see Blumenthal [3], Busemann [2], Pauc [1], Aronszajn-Panitchpakdi [1], and references listed by these authors. A theorem of Whitehead [1] asserts that for any point p of a Riemannian manifold with positive definite metric, all sufficiently small "spherical" neighborhoods of p are metrically convex in a rather strong sense. This result was sharpened by Nijenhuis [1].

Several authors have studied the relationship between the topological structure of a metrizable space and the existence of a convex metric compatible with the given topology. Menger [1] asked whether every continuous curve admits a convex metric, and after several partial results the problem was finally settled affirmatively by Bing [1]. See Bing [1; 2; 3], Plunkett [1], and Lelek-Nitka [1] for this and related results and references.

In order to extend some of his selection theorems, Michael [1] formulated the notion of a *convex structure* on a metric space, describing a situation in which, as in a Riemannian manifold, it is meaningful to form "convex combinations" of certain (but not necessarily all) k-tuples of points in the space. For another metric notion related to convexity, see Michael [2].

Danzer [2; 3; 4] formulates and applies yet another concept of convexity in metric spaces. For each pair x and z of distinct points of M, let $H(x, z) = \{y \in M : \rho(x, y) < \rho(y, z)\}$. Such a set is called a *halfspace*, and a set is called convex provided it is an intersection of halfspaces. This is closely related to Leibniz's idea (see Busemann [1]) of defining a plane as the set of all points equidistant from two given points. From results of Busemann [1] it follows that when (M, ρ) is a normed linear space with metric generated by the norm, then the closed Danzer-convex sets in M are all metrically convex (in the sense of Menger) only when (M, ρ) is an inner-product space. Sets of the form $\{y \in M : \rho(x, y) = \rho(y, z)\}$ are studied also by Kalisch and Straus [1].

Defining the convex hull of a set as the intersection of all halfspaces containing it, Danzer [5] proves that in a metric space, *any family of spherical cells having nonempty intersection covers the convex hull of the set of its centers.* This was the principal tool in his proof of 7.12 above.

9.10. Convexity in topological groups. Though another approach to convexity in topological groups has been formulated by Ghika [4; 5], we consider here only the approach of Rådström [1; 2]. Let G be a topological group and 2^G the class of all nonempty subsets of G. A *one-parameter semigroup of subsets of* G is defined by Rådström [2] to be a biunique mapping A of $]0, \infty[$ into 2^G such that $A(\delta_1 + \delta_2) = A(\delta_1) + A(\delta_2)$ for all $\delta_i \in]0, \infty[$, where the right-hand "+" indicates the usual addition of sets. Rådström [2] proves the following, suggested by the observation that in E^n, "high powers of small sets are in some sense almost convex": *In a locally convex Hausdorff linear space, the one-parameter semigroups of compact sets are exactly those of the form* $A(\delta) = f(\delta) + \delta K$, *where* f *is a one-parameter semigroup of points and* K *is a compact convex set.*

In Rådström's earlier paper [1], a slightly different definition of one-parameter semigroup leads to the conclusion that in R^n, the one-parameter semigroups of sets are exactly those of the form $A(\delta) = \delta K$ for compact convex K. This suggests that in an arbitrary topological group G, one might define convex sets in terms of the one-parameter semigroups, or at least that these semigroups may play a role similar to that played by convex sets in R^n. Rådström [1] shows that if G is a Lie group, then every one-parameter semigroup of subsets of G has, in a sense, an infinitesimal generator which is a convex set.

9.11. Complex convexity. We turn finally to the generalized convexity which plays such an important role in the theory of functions of several complex variables. This generalizes simultaneously the notions of convex set and convex function, with striking parallelism between the real and complex theories. (Of course they differ in details of proof, the complex theory being much more difficult in some respects.) See Bremermann [1; 2; 3] for an excellent description of this parallelism and for references to other work in the field. In particular, Bremermann [1] supplies a "real-complex dictionary" with the following translation of terms:

Real

linear function of one real variable
convex function of one real variable
convex function of n real variables
convex region in R^n

Complex

harmonic function of one complex variable
subharmonic function of one complex variable
plurisubharmonic function of n complex variables
region of holomorphy in C^n (the space of n complex variables).

A real-valued function V on a region $D \subset C^n$ is *plurisubharmonic* provided $-\infty \leqq V < \infty$, V is upper semicontinuous on D, and for every analytic plane $P(z, a) = \{z + \lambda a : \lambda \in C^1\}$ (where $z, a \in C^n$) it is a subharmonic function of λ on the set $D \cap P(z, a)$. A complex-valued function on a region $D \subset C^n$ is *holomorphic* if it is single-valued and is holomorphic in each of the n complex variables separately. The region D is a *region of holomorphy* if there exists a function holomorphic in D and not extendable to a function holomorphic in a larger region. Bremermann [1] justifies the above translations by an impressive list of parallel definitions and theorems, of which the following are samples:

a convex [plurisubharmonic] function assumes its maximum only at the boundary of a domain unless it is constant in the domain;

a domain $D \subset R^n$ [$D \subset C^n$] is convex [a region of holomorphy] if and only if $-\log \delta_D$ is a convex [plurisubharmonic] function in D. (Here δ_D is the inner distance function of D, so that $\delta_D P$ is the radius of the largest Euclidean sphere which is centered at p and contained in D.)

As might be guessed from the above parallelism, the plurisubharmonic functions and the regions of holomorphy are also called *pseudoconvex functions* and *pseudoconvex regions*. (The pseudoconvex regions were originally defined in a different way which turned out to be equivalent to regions of holomorphy.) It would be most interesting to describe axiomatically a structure consisting of certain sets and of functions defined over them in such a way that the theories of real and complex convexity are subsumed under a common generalization. For ideas which may be useful in this connection, see Bauer [**1**; **2**; **3**] and some of his references.

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INDEX OF NOTATION

The general notation is described on pages 105-106. Much of the special notation is used only on the page where it is defined. Important exceptions are listed below, with numbers indicating where the definitions appear.

INDEX OF TERMS